

Exit times, overshoot and undershoot for a surplus process in the presence of an upper barrier

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Abstract

We study the movement of a surplus process with initial capital u in the presence of two barriers: a lower barrier at zero and an upper barrier at b ($b > u$). More specifically, we consider the behaviour of the surplus: (a) in continuous time; and (b) only at claim arrival times. For each of these cases, we find the expected time until the process exits the interval $[0, b]$. We also obtain results related to the undershoot and overshoot of the surplus which, in particular for case (b) above, are derived under the assumption that the distribution of claim sizes and/or claim interarrival times belongs to the mixed Erlang class. In the final section we discuss the implementation of the methods in a number of examples using computer algebra software. These examples illustrate the efficiency of the methods even in fairly complicated cases.

Keywords: Ruin probability; Sparre Andersen model; Barrier problems; Overshoot; Undershoot; Exit times; Mixed Erlang class.

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1 Introduction and Definitions

Much of the research in actuarial risk theory is concerned with ruin problems, where the focus is on the event that the surplus of an insurance company in a certain portfolio becomes negative at some time instant. It has been argued, see for instance Gerber (1990), that in several occasions the event of ruin may not be of primary concern (e.g. when the probability of ruin is very small) but the insurer is rather interested in the time it takes until the surplus reaches a given target. We denote this target surplus by b . For the classical model with Poisson arrivals, Gerber (1990) studied the time, $\tau_{u,b}$, that it takes until the surplus process reaches the target surplus b when the initial surplus is u , regardless of whether ruin occurs prior to that time, and obtained the moment generating function (m.g.f.) of $\tau_{u,b}$, so that the associated moments are explicitly available. More recently, Dickson and Li (2013) gave the corresponding result for the case where the interarrival times between claims have an Erlang (2) distribution. This is a special case of the renewal (Sparre Andersen) risk model, which we define below. In contrast with the two papers mentioned above, here we consider both the upward and the downward movement of the surplus process, so that we study the risk model with two barriers, a lower barrier at zero and an upper barrier at b . More specifically, we consider two cases: the first (Case A) concerns the continuous-time movement of the surplus process. For this case, we obtain an expression for the mean time until the process exits the interval $[0, b]$. Case B deals with the movement of the process *at claim instants*. For this latter case, and assuming that both claim sizes and the claim interarrival times are mixed Erlang, we obtain explicit expressions for the distribution of the deficit at ruin (essentially, the undershoot of the surplus process) and the distribution of the overshoot above the barrier b .

Problems similar to those considered in Case A have been studied earlier by Wang and Politis (2002) for the classical model, Zhou (2004) for the classical model with diffusion and Yang and Zhang (2010) for a Markov modulated model. Kyprianou (2006, Ch.8) discusses exit problems for spectrally negative Lévy processes. In Case B, and further specialising to the classical risk model (exponential interarrival times) with no upper barrier, our results in Section 3.2, although apparently in a different form, agree with the results obtained by Willmot and Woo (2007) (see the numerical example in Section 3.2). Although there exist a number of theoretical results related to transforms of exit times (see, e.g. Kyprianou (2006) and Jacobsen (2011)), explicit calculations for quantities related to exit times is a non-trivial task and we aim to address it in the present paper (see, in particular, the last section).

We first introduce some notation for the Sparre Andersen model. The surplus process is $\{U(t) : t \geq 0\}$, where the insurer's surplus at time t is given by

$$U(t) := u + ct - \sum_{i=1}^{N(t)} X_i.$$

Here X_i are the independent and identically distributed (i.i.d.) sizes of claim amounts, c is the premium income rate, F is the distribution function of the X_i , $u = U(0)$ is the initial surplus and $N(t)$ represents the number of claims up to time t . The mean size of a claim is denoted by μ . The times between successive claims $T_1, T_2, \dots$, are also assumed i.i.d. random variables with mean λ^{-1} . We write K for the common distribution of the T_i and we note that $\{N(t) : t \geq 0\}$ is a renewal process. In the special case where the distribution of the T_i for $i = 1, 2, \dots$, is exponential, then $\{N(t) : t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and we talk about the classical model.

As mentioned already, here we consider the process $U(t)$ with a lower barrier at zero and an upper barrier at some $b > u$. Figure 1 shows two paths of the surplus process; in the graph on the left, the process hits b before ruin, while on the right graph ruin occurs before reaching b . (Of course, there is a third possibility, not shown in the figure, that ruin does not occur.)

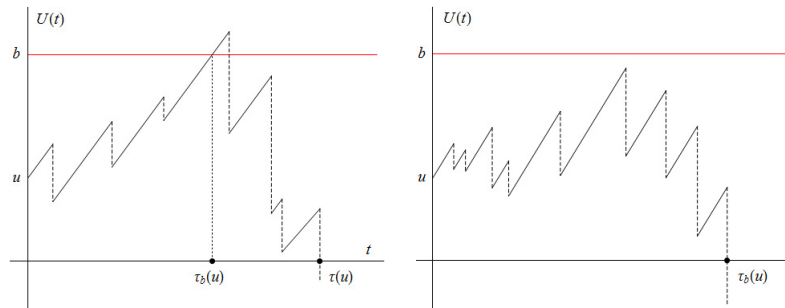


Figure 1: *The surplus process with two barriers.*

The outline of the paper is as follows: Section 2 sets up the necessary background and presents some preliminary results for the model with two barriers. In the same section we obtain the expectation of the time until the continuous-time process (Case A) exits the interval $[0, b]$. Section 3 deals with Case B, using a discrete-time setup. For this case, and in order to obtain explicit results we concentrate on the case where one or both of the distributions F and K belong to a large class of distributions, namely the mixed Erlang class. More specifically, in Section 3.1 we obtain analytic results for a number of ruin quantities assuming that both F and K are mixed Erlang. As a byproduct of the analysis, we derive the conditional distribution for the deficit at ruin (given that ruin occurs) for the classical model, when no upper barrier exists (cf. Section 3.2). The examples in the last section illustrate that, although

the calculations needed for the various results are often fairly complicated, they are easily and efficiently carried out using computer algebra software.

2 Ruin probabilities

2.1 Preliminaries and notation

Let $\psi(u)$ be the probability of ruin in the usual case with only one barrier at zero,

$$\psi(u) := P(U(t) < 0 \text{ for some } t \geq 0 | U(0) = u) = P(\tau(u) < \infty),$$

where $\tau(u)$ is the time until ruin occurs (a defective random variable). However, the time $\tau_{u,b}$ to reach the given target,

$$\tau_{u,b} := \inf\{t : U(t) \geq b\} = \inf\{t : U(t) = b\},$$

is a proper random variable under the assumption that $c > \lambda\mu$. We now introduce a quantity of primary interest in the present paper, the time that the surplus process exits the interval $[0, b]$, namely

$$\tau_b(u) := \inf\{t : U(t) \notin [0, b]\}.$$

It is clear that $\tau_b(u)$ is a stopping time for $\{U(t) : t \geq 0\}$ which is finite with probability one, i.e. $P(\tau_b(u) < \infty) = 1$; moreover, $\tau_b(u) = \min\{\tau(u), \tau_{u,b}\}$. Gerber and Shiu (1998) studied a similar problem to ours, by considering the minimum between $\tau_{u,b}$ and the time of *upcrossing the value zero after ruin*, if ruin occurs. Here we consider a related (but different) problem; however, as in the case of Gerber and Shiu (1998), the results of the present paper can be useful in studying the distribution of the surplus prior to ruin, see also Dickson (1992).

Let now $\psi_b(u)$ be the probability that ruin occurs and this happens before the process has reached the barrier b . In other words, $\psi_b(u)$ is the probability of the event $\{\tau_b(u) \neq \tau_{u,b}\}$. First, we express $\psi_b(u)$ in terms of $\psi(u)$ as follows. Conditioning on whether ruin occurs before or after hitting the barrier b , we get

$$\psi(u) = \psi_b(u) + (1 - \psi_b(u)) \phi_u(b), \quad (2.1)$$

where $\phi_u(b) := P(\text{the process ruins, given that it starts from } u \text{ and hits } b \text{ before falling below } 0)$. In the classical model, due to the memoryless property of the interarrival times, we get that $\phi_u(b) = \psi(b)$, so that

$$\psi_b(u) = \frac{\psi(u) - \psi(b)}{1 - \psi(b)}, \quad b > u \geq 0 \quad (2.2)$$

or, equivalently,

$$\delta_b(u) = \frac{\delta(u)}{\delta(b)}, \quad (2.3)$$

where $\delta_b(u) := 1 - \psi_b(u)$ and $\delta(u) := 1 - \psi(u)$ are the survival probabilities for the two- and one-barrier model respectively; see Dickson and Gray (1984). Thus, in the classical model, the probability $\psi_b(u)$ for the model with two barriers is available whenever we have an expression for the probability $\psi(u)$ in the usual, one-barrier case.

2.2 The renewal model

In the general (renewal) case, an expression for $\phi_u(b)$ can be obtained by conditioning on the amount of the overshoot above the barrier b in the unrestricted process. More specifically, let U_b be the value of the surplus when the first claim, after crossing the barrier, arrives (given that surplus hits the upper barrier first, i.e. that $\tau_b(u) = \tau_{u,b}$). Then $U_b - b$ is the overshoot and we define a new variable $W_b := (U_b - b)/c$, which we call the *overshoot time*. Let G_u be the distribution function of W_b . Denote also by X the first claim amount that arrives after the overshoot. Conditioning on the values of X and W_b we get that

$$\phi_u(b) = \int_0^\infty \int_0^{b+cx} \psi(b+cx-y) dF(y) dG_u(x) + \int_0^\infty \int_{b+cx}^\infty 1 dF(y) dG_u(x). \quad (2.4)$$

Note that the sum of the two integrals on the right can be expressed in compact form as $E(\psi(b+cW_b-X))$, assuming that $\psi(u) = 1$ for $u < 0$. Therefore, (2.1) yields that

$$\delta_b(u) = \frac{\delta(u)}{E(\delta(b+cW_b-X))}, \quad (2.5)$$

which has a similar form with (2.3).

Next, in view of (2.1) and (2.4), if the probability of ruin $\psi(u)$ is known, in order to determine the probability $\psi_b(u)$ the only unknown quantity is the distribution G_u of the overshoot time. For example, if the distribution K of the interarrival times T_i is exponential (classical model), then G_u is also exponential.

A useful general expression for the distribution of the overshoot is given in the following lemma. Before stating this lemma we need some notation. Denote by T_b^* the time interval between the successive claim arrivals that includes the crossing time $\tau_{u,b}$ (given that $\tau_b(u) = \tau_{u,b}$). Then $V_b := T_b^* - W_b$ is the time between the crossing time $\tau_b(u)$ and the last claim arrival time before $\tau_b(u)$.

Lemma 1. *The distribution tail of the overshoot time (given that $\tau_b(u) = \tau_{u,b}$) can be expressed as*

$$1 - G_u(y) = P(W_b > y) = \int_0^\infty \frac{\bar{K}(x+y)}{\bar{K}(x)} dF_{b,u}(x), \quad (2.6)$$

for every $y \geq 0$, where $F_{b,u}$ denotes the c.d.f. of V_b .

Proof. We observe that, for $x, y \geq 0$, the probability $P(T_b^* > x+y | T_b^* = T_n, V_b = x)$ is equal to

$$\begin{aligned} & P\left(T_n > x+y | U\left(\sum_{i=1}^{n-1} T_i\right) = b-cx, \text{ and } U(t) < b \text{ for all } t < \sum_{i=1}^{n-1} T_i, \text{ and } T_n > x\right) \\ &= P(T_n > x+y | T_n > x) = P(T_1 > x+y | T_1 > x) \end{aligned}$$

and therefore,

$$\begin{aligned} P(T_b^* > x+y | V_b = x) &= \sum_{n=1}^{\infty} P(T_b^* > x+y | T_b^* = T_n, V_b = x) P(T_b^* = T_n | V_b = x) \\ &= \sum_{n=1}^{\infty} P(T_1 > x+y | T_1 > x) P(T_b^* = T_n | V_b = x) \\ &= P(T_1 > x+y | T_1 > x). \end{aligned}$$

It follows that, for every $x, y \geq 0$,

$$P(W_b > y | V_b = x) = P(T_b^* > x + y | V_b = x) = P(T_1 > x + y | T_1 > x) = \frac{\bar{K}(x + y)}{\bar{K}(x)}$$

which immediately leads to (2.6). $\square$

It is worth mentioning that a result similar to the above lemma can also be derived for the undershoot (deficit at ruin).

In general, the distribution of the overshoot cW_b is very difficult to determine. However, in cases where the distribution of T_i can be expressed via exponential distributions (e.g. as a mixture), we can employ the memoryless property (or, more formally, (2.6)) and deduce a representation for G_u .

Example 1. Suppose that K is a mixture of n exponential distributions, so that the associated tail can be written in the form

$$\bar{K}(x) = \sum_{i=1}^n p_i e^{-\lambda_i x}, \quad x \geq 0, \quad \sum_{i=1}^n p_i = 1$$

where $0 < p_i < 1$ for $i = 1, 2, \dots, n$. Using (2.6) we now get that, for $y \geq 0$,

$$\bar{G}_u(y) = \int_0^\infty \frac{\bar{K}(x + y)}{\bar{K}(x)} dF_{b,u}(x) = \sum_{i=1}^n e^{-\lambda_i y} \int_0^\infty \frac{p_i e^{-\lambda_i x}}{\bar{K}(x)} dF_{b,u}(x) = \sum_{i=1}^n a_i e^{-\lambda_i y}$$

where $a_i := p_i E(\bar{K}(V_b)^{-1} e^{-\lambda_i V_b}) \in (0, 1)$. Hence the distribution of the overshoot is again a mixture of the same exponential distributions with different weights.

Dickson (1998) has pointed out that (2.2) does not hold for the renewal risk model. The point here is that, in the compound Poisson model and due to the memoryless property of interarrival times, once the surplus reaches b for the first time (before ruin), the process starts afresh probabilistically, so that $\phi_u(b) = \psi(b)$, see the argument leading to (2.2). In the general case, when the surplus process reaches b , we do not know the amount of time since the previous claim.

In view of the above, we recall the definition of the following well-known classes of life distributions. A distribution $F(y) = 1 - \bar{F}(y)$ of a nonnegative random variable is said to be new better (worse) than used (NBU, resp. NWU) if $\bar{F}(x + y) \leq (\geq) \bar{F}(x)\bar{F}(y)$ for all $x, y \geq 0$. We then have the following result.

Proposition 2. *In the renewal risk model, if the distribution K of interarrival times is NBU, then*

$$\delta_b(u) \geq \frac{\delta(u)}{\delta(b)} \quad (2.7)$$

while if K is NWU the sign of the inequality is reversed.

Proof. Since K is NBU we have that $\bar{K}(x + y) \leq \bar{K}(x)\bar{K}(y)$ and thus, using (2.6),

$$1 - G_u(y) = P(W_b > y) = \int_0^\infty \frac{\bar{K}(x + y)}{\bar{K}(x)} dF_{b,u}(x) \leq \int_0^\infty \bar{K}(y) dF_{b,u}(x) = \bar{K}(y),$$

which implies that W_b is stochastically smaller than a r.v. T where $T \sim K$. Since δ is increasing, using (2.5) we get that

$$\delta_b(u) = \frac{\delta(u)}{E(\delta(b + cW_b - X))} \geq \frac{\delta(u)}{E(\delta(b + cT - X))} = \frac{\delta(u)}{\delta(b)}.$$

In the case where K is NWU the inequalities are reversed, and this completes the proof. $\square$

We note that the bound $\delta(u)/\delta(b)$ for $\delta_b(u)$ in Proposition 2 can be found using the results in Landriault and Willmot (2008), when K is arbitrary and F is a Coxian distribution; see also Willmot (2007).

Example 2. We illustrate numerically the performance of the inequality (2.7) in two cases:

(i) If T_i 's follow an Erlang(2) distribution with scale parameter l , then K is NBU. Hence, according to Proposition 2, $\delta_b(u) \geq \delta(u)/\delta(b)$.

(ii) If T_i 's follow a Pareto distribution with density $\kappa(x) = a_0 \lambda_0^{a_0} (x + \lambda_0)^{-a_0-1}$, $x \geq 0$, then K is NWU and thus $\delta_b(u) \leq \delta(u)/\delta(b)$.

The inequalities in the above two cases are numerically verified in the following table (up to 4 digits) for several values of the parameters. We assume that $b = 6, u = 2, c = 1$ and in (i) X_i 's follow an Erlang(2) with scale parameter 2 while in (ii) X_i 's follow a mixed exponential distribution with pdf $f(x) = \frac{1}{5}e^{-x/3} + \frac{3}{50}e^{-x/5} + \frac{1}{200}e^{-x/20}$, $x > 0$.

l	$\delta_b(u)$	$\delta(u)/\delta(b)$	λ_0, a_0	$E(T), V(T)$	$\delta_b(u)$	$\delta(u)/\delta(b)$
1	0.9463	0.9457	20, 3	10, 300	0.7754	0.8014
1.7	0.6534	0.6411	12, 3	6, 108	0.6439	0.7026
2.4	0.1613	0.1402	30, 6	6, 54	0.6926	0.7180

(i)

(ii)

The values of $\delta(u), \delta(b)$ in (ii) were computed via the method of Landriault and Willmot (2008) (cf. the example on pages 607-608 in that paper).

Next, we consider the following quantity M , namely the maximal severity of ruin, introduced by Picard (1994) and studied also by Dickson (1998) in a context similar to ours, i.e. in the presence of an upper barrier. In words, the maximal severity denotes the largest magnitude of the deficit between the time of ruin and the first time that the surplus process crosses upwards level 0. Recall that $\tau(u)$ is the time of ruin and define $\tau^{(1)}(u) := \inf\{t : t > \tau(u) \text{ and } U(t) > 0\}$. The maximal severity of ruin (given that ruin occurs) is then defined as

$$M := \sup\{|U(s)|, \tau(u) \leq s \leq \tau^{(1)}(u)\}, \quad (2.8)$$

and the associated distribution function is $J_u(x) := P(M \leq x | U(0) = u, \tau(u) < \infty)$. Dickson (1998) finds an approximation for this quantity for the case where an upper barrier exists. Following a similar approach and employing the result of Proposition 2 we easily get the following inequality.

Proposition 3. *If the distribution K is NBU (NWU), then*

$$J_u(x) \geq (\leq) \frac{\delta(u+x) - \delta(u)}{(1 - \delta(u))\delta(x)}.$$

Proof. If $F_{de,u}$ denotes the c.d.f. of the deficit at ruin, it is easy to see that

$$\delta(u+x) = \delta(u) + (1 - \delta(u)) \int_0^x \delta(x-y) dF_{de,u}(y).$$

From the above and Proposition 2 we readily get that

$$J_u(x) = \int_0^x \delta_x(x-y) dF_{de,u}(y) \geq \int_0^x \frac{\delta(x-y)}{\delta(x)} dF_{de,u}(y) = \frac{\delta(u+x) - \delta(u)}{\delta(x)(1 - \delta(u))}$$

The proof for the NWU case is the same, with the inequality sign reversed. $\square$

2.3 The deficit at ruin and the time to reach the barrier in the classical model

In this section we obtain a result for the mean time, $E(\tau_b(u))$, until the surplus exits the interval $[0, b]$. Let $G(u, y)$ be the probability that ruin occurs and the deficit at the time of ruin is at most y , when no upper barrier exists, i.e.

$$G(u, y) := P(\tau(u) < \infty, |U(\tau(u))| \leq y).$$

Then this is a (defective) distribution function; see, for instance, Gerber et al. (1987), Willmot (2002), and Psarrakos (2008) for a discussion and various properties of this function. Considering now the surplus process with two barriers, at 0 and b , let $G_b(u, y)$ be the probability that ruin occurs *before the process crosses the upper barrier* and that the deficit at the time of ruin is at most y . For the classical risk model, we get the following result which generalises (2.2).

Proposition 4. *In the classical risk model it holds that*

$$G_b(u, y) = G(u, y) - G(b, y) \cdot \frac{\delta(u)}{\delta(b)}, \quad b > u \geq 0 \quad (2.9)$$

Proof. Considering whether ruin occurs before or after the surplus hits the upper barrier, we obtain that $G(u, y) = G_b(u, y) + \delta_b(u)G(b, y)$, $b > u \geq 0$, and the result follows solving for $G_b(u, y)$ and using (2.3). $\square$

Therefore, $G_b(u, y)$ is available whenever $G(u, y)$ is known. This includes in particular the class of mixed Erlang distributions, see Willmot and Woo (2007). Define now the functions $\bar{G}(u, y) := \psi(u) - G(u, y)$, $u, y \geq 0$ and

$$\bar{G}_b(u, y) := \psi_b(u) - G_b(u, y), \quad b > u \geq 0, y \geq 0. \quad (2.10)$$

With this notation, $\bar{G}(u, y)$ denotes the probability that ruin occurs and that the deficit at the time of ruin exceeds y , with a similar interpretation for $\bar{G}_b(u, y)$. We are now ready to state the following result.

Proposition 5. *In the classical risk model,*

$$E(\tau_b(u)) = \frac{\delta(u)}{\delta(b)} \cdot \frac{b + \int_0^\infty \bar{G}(b, y) dy}{c - \lambda\mu} - \frac{u + \int_0^\infty \bar{G}(u, y) dy}{c - \lambda\mu}. \quad (2.11)$$

Proof. For convenience, consider the *net profit process* $\{B(t) : t \geq 0\}$ where $B(t) := U(t) - u$, $t \geq 0$. For the classical model, with Poisson arrival rate λ and mean claim size μ , we know that (see Wang and Politis (2002))

$$E(\tau_b(u)) = \frac{E(B(\tau_b(u)))}{c - \lambda\mu}$$

and $E(B(\tau_b(u))) = (b - u)(1 - \psi_b(u)) + (-u + E(U(\tau_b(u)) | \tau(u) = \tau_b(u)))\psi_b(u)$. Let now

$$\bar{C}(y) := \frac{\bar{G}_b(u, y)}{\psi_b(u)} = 1 - \frac{G_b(u, y)}{\psi_b(u)}, \quad u, y \geq 0.$$

Then $C(y) = 1 - \bar{C}(y)$ is a proper distribution function which expresses the probability that, given that ruin occurs before hitting the barrier, the amount of the deficit is at most y . Then

$$E(-U(\tau_b(u)) | \tau(u) = \tau_b(u)) = \int_0^\infty \bar{C}(y) dy.$$

Putting all the above results together, we obtain that

$$E(\tau_b(u)) = \frac{(b-u)\delta_b(u) - \psi_b(u)(u + \int_0^\infty \bar{C}(y) dy)}{c - \lambda\mu} = \frac{b\delta_b(u) - u - \int_0^\infty \bar{G}_b(u, y) dy}{c - \lambda\mu}$$

The above expression now easily leads to (2.11) by substituting $\delta_b(u)$ from (2.3) and taking into account the equality $\bar{G}_b(u, y) = \bar{G}(u, y) - \bar{G}(b, y)\delta_b(u)$. $\square$

Regarding the term $\int_0^\infty \bar{G}(u, y) dy$ which appears in the last proposition, we note that this can be written in the form

$$\int_0^\infty \bar{G}(u, y) dy = \psi(u)E(-U(\tau(u)) \mid \tau(u) < \infty, U(0) = u). \quad (2.12)$$

Explicit results for the last quantity on the right (the expectation of the deficit, given that ruin occurs) can be found in Lin and Willmot (2000).

3 The surplus process on claim arrival times, in the presence of an upper barrier

In this section we consider again the renewal risk model, but this time we assume that the insurer's surplus is inspected only at claim arrival times. More specifically, we assume that the surplus process terminates at the first time it passes below zero or is above the barrier b immediately after the payment of a claim. From a practical viewpoint, it might be more appropriate to have this modified stopping rule, since in the previous case when the process hits b in its upward movement, a (potentially large) claim might be imminent. This modification will enable us to obtain explicit results for ruin-related quantities by relying on an appropriate discrete time model. This model is built upon the observation that the process now necessarily stops at claim arrival times $R_k := \sum_{i=1}^k T_i, k = 1, 2, \dots$ and therefore it is sufficient to concentrate on the discrete-time movement of the random walk $S := \{S_k : k = 0, 1, 2, \dots\}$ where

$$S_k := U(R_k) = u + cR_k - \sum_{i=1}^k X_i = u + \sum_{i=1}^k (cT_i - X_i) = u + \sum_{i=1}^k Z_i, \quad k = 1, 2, \dots$$

and $Z_i := cT_i - X_i$. The first step that the process S falls below zero or exceeds the upper barrier b is denoted by

$$\nu = \nu_b(u) := \inf\{k : S_k \notin [0, b]\}.$$

Obviously, $\nu_b(u)$ is a stopping time associated with the process $\{S_k : k = 1, 2, \dots\}$ and in the sequel we shall use the simplified notation ν (although ν depends on b, u). The associated exit time will be denoted by $\tilde{\tau}_b(u) := R_\nu$ and it is clear that $\tilde{\tau}_b(u) \geq \tau_b(u)$.

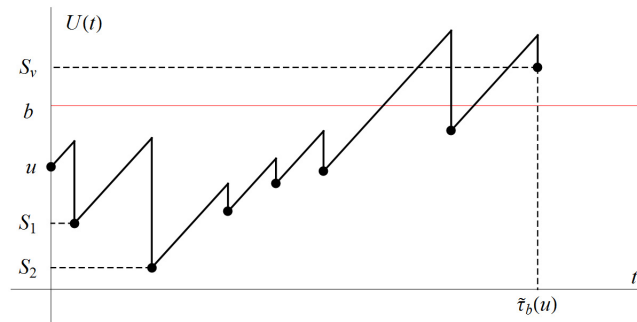


Figure 2 : The surplus process at claim arrival times

We are interested in the probability of ruin before the discrete time process S exceeds b , i.e.

$$\tilde{\psi}_b(u) := 1 - \tilde{\delta}_b(u) := P(S_\nu < 0).$$

Recall that $G_b(u, y) := P(\tau(u) = \tau_b(u), |U(\tau(u))| \leq y)$ denotes the probability that ruin occurs before the process U crosses b and the deficit is at most y . Similarly, denote by $\tilde{G}_b(u, y)$ the probability that ruin occurs before the process S crosses b and the deficit is at most y . Also denote by $\tilde{F}_{ov}$ the c.d.f. of the overshoot $S_\nu - b$, given that $S_\nu > b$, and by $\tilde{F}_{de}$ the c.d.f. of the deficit $-S_\nu$, given that $S_\nu < 0$ (we use this simplified notation although these c.d.f.'s depend on b, u). Note that

$$E(\delta(S_\nu) | S_\nu > b) = \frac{1}{1 - F_{S_\nu}(b)} \int_b^\infty \delta(t) dF_{S_\nu}(t) \stackrel{x=t-b}{=} \int_0^\infty \delta(x+b) d\tilde{F}_{ov}(x),$$

where F_{S_ν} is the c.d.f. of S_ν . Similarly, we have that $E(G(S_\nu, y) | S_\nu > b) = \int_0^\infty G(x+b, y) d\tilde{F}_{ov}(x)$. In the renewal model we obtain the following proposition which relates $\tilde{\psi}_b(u)$ with $\psi(u)$ and $\tilde{G}_b(u, y)$ with $G(u, y)$.

Proposition 6. *The probabilities $\tilde{\delta}_b(u)$ and $\tilde{G}_b(u, y)$ can be expressed as follows*

$$\begin{aligned} (a) \quad \tilde{\delta}_b(u) &= \frac{\delta(u)}{E(\delta(S_\nu) | S_\nu > b)}, \\ (b) \quad \tilde{G}_b(u, y) &= G(u, y) - E(G(S_\nu, y) | S_\nu > b) \tilde{\delta}_b(u). \end{aligned}$$

Proof. Set $A_y := [\tau(u) < \infty, |U(\tau(u))| \leq y]$. Conditioning on the mutually exclusive events $[S_\nu < 0]$ and $[S_\nu \geq b]$, we have that

$$\begin{aligned} G(u, y) &= P(A_y) = \tilde{\delta}_b(u) P(A_y | S_\nu \geq b) + P(-S_\nu \leq y, S_\nu < 0) \\ &= \tilde{\delta}_b(u) \int_0^\infty P(A_y | S_\nu - b = x) d\tilde{F}_{ov}(x) + P(-S_\nu \leq y, S_\nu < 0) \\ &= \tilde{\delta}_b(u) \int_0^\infty G(b+x, y) d\tilde{F}_{ov}(x) + \tilde{G}_b(u, y), \end{aligned}$$

which leads to (b). By taking $y \rightarrow \infty$ in the above relation we obtain immediately that

$$1 - \delta(u) = \tilde{\delta}_b(u) \int_0^\infty (1 - \delta(b+x)) d\tilde{F}_{ov}(x) + 1 - \tilde{\delta}_b(u),$$

which establishes (a). $\square$

It is also evident that $\tilde{G}_b(u, y) = P(-S_\nu \leq y, S_\nu < 0) = \tilde{F}_{de}(y) \tilde{\psi}_b(u)$. Analogously to (2.8), the maximal severity of ruin (given that ruin occurs) considered only on claim arrival times is

$$\tilde{M} := \sup\{|S_k|, \nu \leq k < \nu^{(1)}\}, \quad \nu^{(1)} = \inf\{k : k > \nu \text{ and } S_k > 0\}$$

and the associated distribution function is

$$\begin{aligned} \tilde{J}_u(x) &:= P(\tilde{M} \leq x | S_\nu < 0) = \int_0^\infty P(\tilde{M} \leq x | S_\nu < 0, -S_\nu = y) d\tilde{F}_{de}(y) \\ &= \int_0^\infty \tilde{\delta}_x(x-y) d\tilde{F}_{de}(y) = E(\tilde{\delta}_x(x+S_\nu) | S_\nu < 0) \end{aligned} \tag{3.1}$$

noting that we have defined $\tilde{\delta}_b(u) = 0$ when $u < 0$. From the above and in view of (a) in Proposition 6, we also get that

$$\tilde{J}_u(x) = \frac{E(\delta(x+S_\nu) | S_\nu < 0)}{E(\delta(S_\nu) | S_\nu > b)}, \quad x \geq 0.$$

3.1 Mixed Erlang claims and interarrival times

In this subsection we present the main result for the discrete time model when an upper barrier exists. More explicitly when the distributions K and F belong to the *mixed Erlang class* of distributions we are able to obtain explicit results for the ruin probability and the distributions of the overshoot $\tilde{F}_{ov}$ and the deficit $\tilde{F}_{de}$. The mixed Erlang class has been studied by Willmot and Lin (2011), who discussed several properties and its significance in an actuarial context. One such property is that this class is wider than the class of phase-type distributions. The c.d.f. of a mixed Erlang distribution has the following form

$$F_{\mathcal{ME}}(x) := \sum_{k=1}^{\infty} h_k F_{\mathcal{E}_{k,\lambda}}(x), \quad x \geq 0,$$

where $h_k \in [0, 1]$, $k = 1, 2, \dots$, and $\sum_{k=1}^{\infty} h_k = 1$, while $F_{\mathcal{E}_{k,\lambda}}$ denotes the c.d.f. of an Erlang distribution with shape parameter k , scale parameter λ . The importance of this class follows from the facts that (a) it is dense in the class of all non-negative distributions (see Remark 1), (b) it possesses a “*generalized form of the memoryless property*” (see Lemma 7).

Remark 1. Any distribution H on $[0, \infty)$ can be approximated by the following mixture of Erlang distributions,

$$H_n(x) := H(0) + \sum_{k=1}^{\infty} p_{k,n} F_{\mathcal{E}_{k,n}}(x), \quad x \geq 0, \quad \text{where } p_{k,n} := H\left(\frac{k}{n}\right) - H\left(\frac{k-1}{n}\right). \quad (3.2)$$

Note that the sequence $H_n, n = 1, 2, \dots$ converges weakly to H (cf. Schassberger (1970)). In practical applications, though, one may prefer to work with a finite summation in (3.2), e.g. for $\sum_{k=1}^m$, considering that the class of finite mixtures of Erlang distributions, for all $m = 1, 2, \dots$, is also dense in the class of non-negative distributions.

In what follows, we assume that the common distribution F of X_i ’s and the common distribution K of T_i ’s are finite mixtures of Erlang distributions of the form

$$F(x) := \sum_{k=1}^{m_1} d_k F_{\mathcal{E}_{k,1/\alpha}}(x), \quad K(x) := \sum_{k=1}^{m_2} h_k F_{\mathcal{E}_{k,\beta}}(x), \quad x \geq 0, \quad (3.3)$$

where $\sum_{k=1}^{m_1} d_k = 1$, ($d_{m_1} \neq 0$) and $\sum_{k=1}^{m_2} h_k = 1$, ($h_{m_2} \neq 0$).

The mixed Erlang distribution can be seen as a random sum of exponentially distributed r.v.’s and hence we may say that it possesses a “generalized form of the memoryless property”. This is formally expressed in the next lemma.

Lemma 7. *If the distribution K of the interarrival times is a mixed Erlang distribution (i.e. has the form (3.2)) then the overshoot time W_b (given that $\tau_b(u) = \tau_{u,b}$) follows also a mixed Erlang distribution (in general, with a different mixing distribution).*

Proof. Denote by κ the p.d.f. associated with K . Using (2.6) we get that, for $y \geq 0$, the p.d.f. g of G can be expressed as follows

$$\begin{aligned} g(x) &= \int_0^\infty \frac{\kappa(x+y)}{\bar{K}(x)} dF_b(x) = \int_0^\infty \frac{\sum_{i=1}^\infty \frac{h_i \lambda^i}{(i-1)!} (x+y)^{i-1} e^{-\lambda(x+y)}}{\bar{K}(x)} dF_b(x) \\ &= \int_0^\infty \frac{\sum_{i=1}^\infty \frac{h_i \lambda^i}{(i-1)!} \sum_{j=0}^{i-1} \binom{i-1}{j} y^j x^{i-1-j} e^{-\lambda(x+y)}}{\bar{K}(x)} dF_b(x) = \sum_{j=1}^\infty a_j \frac{\lambda^j y^{j-1} e^{-\lambda y}}{(j-1)!} \end{aligned}$$

where

$$a_j = \sum_{i=j}^\infty h_i \frac{\lambda^{i-j}}{(i-j)!} \int_0^\infty \frac{x^{i-j} e^{-\lambda x}}{\bar{K}(x)} dF_b(x) \in (0, 1), \quad j = 1, 2, \dots$$

The interchange of the integral and the summation in the third step above is valid by invoking Tonelli's theorem. Hence the distribution of the overshoot is again a mixture of Erlang distributions with weights $a_1, a_2, \dots$. $\square$

It is worth mentioning that, following a similar reasoning, an analogous result to the above can be derived for the undershoot (deficit at ruin). In particular, when F is a mixed Erlang distribution, then the deficit at ruin is also a mixed Erlang distribution. This is a well known result, e.g. see Willmot and Woo (2007).

We are now interested in finding a way to evaluate (i) the probability of ruin $\tilde{\psi}_b(u) = P(S_\nu < 0)$, (ii) the distribution of the deficit at ruin $\tilde{F}_{de}$ (given that ruin occurs) and (iii) the distribution of the overshoot $\tilde{F}_{ov}$ above b (given that the overshoot occurs, i.e. the process hits the upper barrier first). The following theorem offers a representation for these three quantities that can lead to their analytic form (see Examples 3, 4) or their precise computation when the model parameter values are given (see Examples 5, 6). Before stating this theorem we introduce some necessary notations. Denote by $w_1, w_2, \dots, w_m \in \mathbb{C}$ the $m := m_1 + m_2 - 1$ roots of the following polynomial equation w.r.t. w (excluding the root $w = 0$)

$$\sum_{i=1}^{m_2} h_i (1 - \gamma w)^{m_2-i} \sum_{j=1}^{m_1} d_j (1 + \alpha w)^{m_1-j} = (1 + \alpha w)^{m_1} (1 - \gamma w)^{m_2} \quad (3.4)$$

where $\gamma := c/\beta$. Denote also by $\mathbf{B}$ the $m \times m$ matrix with rows $e^{-w_j u} (B_{j1}, B_{j2}, B_{j3})$, $j = 1, 2, \dots, m$ where

$$\begin{aligned} B_{j1} &:= \frac{e^{w_j b}}{(1 - \gamma w_j)^{m_2}} - e^{w_j u}, \\ B_{j2} &:= \left(e^{w_j b} \frac{(1 - \gamma w_j)^{m_2-i} - 1}{(1 - \gamma w_j)^{m_2}}, \quad i = 1, 2, \dots, m_2 - 1 \right) \\ B_{j3} &:= \left(\frac{(1 + \alpha w_j)^{m_1-i} - 1}{(1 + \alpha w_j)^{m_1}}, \quad i = 1, 2, \dots, m_1 - 1 \right) \end{aligned}$$

and by $\mathbf{a}$ the $m \times 1$ vector

$$\mathbf{a} := \left(1 - \frac{e^{-w_1 u}}{(1 + \alpha w_1)^{m_1}}, 1 - \frac{e^{-w_2 u}}{(1 + \alpha w_2)^{m_1}}, \dots, 1 - \frac{e^{-w_m u}}{(1 + \alpha w_m)^{m_1}} \right)^T.$$

In the sequel we shall assume that $\mathbf{B}^{-1}$ exists, which seems a harmless assumption for practical purposes. We denote by $a, \kappa'_1, \dots, \kappa'_{m_2-1}, \pi_1, \dots, \pi_{m_1-1}$ the elements of the $m \times 1$ vector

$$\pi := (a, \kappa'_1, \dots, \kappa'_{m_2-1}, \pi_1, \dots, \pi_{m_1-1})^T := \mathbf{B}^{-1} \cdot \mathbf{a}.$$

As it can be seen in the following theorem, all quantities of interest can be expressed via the elements of the above vector π .

Theorem 8. (i) The ruin probability $\tilde{\psi}_b(u) = P(S_\nu < 0)$ is given by $\tilde{\psi}_b(u) = (1 + a)^{-1}$.
(ii) The distribution $\tilde{F}_{de}$ of the deficit at ruin, $-S_\nu$ (given that ruin occurs before hitting the barrier, i.e. $S_\nu < 0$), can be expressed as a mixture of Erlang distributions as follows,

$$\tilde{F}_{de}(x) = \sum_{i=1}^{m_1} \pi_i F_{\mathcal{E}_{i,1/\alpha}}(x), \quad x \geq 0, \quad \sum_{i=1}^{m_1} \pi_i = 1,$$

(iii) The distribution $\tilde{F}_{ov}$ of the overshoot $S_\nu - b$ (given that $S_\nu > b$) can be expressed as a finite mixture of Erlang distributions as follows,

$$\tilde{F}_{ov}(x) = \sum_{i=1}^{m_2} \kappa_i F_{\mathcal{E}_{i,1/\gamma}}(x), \quad x \geq 0, \quad \sum_{i=1}^{m_2} \kappa_i = 1,$$

where $\gamma = c/\beta$ and $\kappa_i := \kappa'_i/a, i = 1, 2, \dots, m_2$.

Proof. The p.d.f. of $Z_i = cT_i - X_i$ is given by the following expression

$$\begin{aligned} f_Z(x) &= \int_{\max\{-x, 0\}}^{\infty} \frac{1}{c} f_T\left(\frac{x+t}{c}\right) f_X(t) dt \\ &= \int_{\max\{-x, 0\}}^{\infty} \frac{1}{c} \sum_{i=1}^{m_2} \frac{h_i \beta^i}{(i-1)!} \left(\frac{x+t}{c}\right)^{i-1} e^{-\beta \frac{x+t}{c}} \sum_{k=1}^{m_1} \frac{d_k \alpha^{-k}}{(k-1)!} t^{k-1} e^{-\frac{t}{\alpha}} dt. \\ &= \sum_{i=1}^{m_2} \sum_{k=1}^{m_1} \frac{h_i d_k \gamma^{-i} \alpha^{-k} e^{-\frac{x}{\gamma}}}{(i-1)!(k-1)!} \int_{\max\{-x, 0\}}^{\infty} (x+t)^{i-1} t^{k-1} e^{-(\frac{1}{\alpha} + \frac{1}{\gamma})t} dt. \end{aligned}$$

Now, since the above integral has the following representation,

$$\int_{\max\{-x, 0\}}^{\infty} (x+t)^{i-1} t^{k-1} e^{-(\frac{1}{\alpha} + \frac{1}{\gamma})t} dt = \begin{cases} \sum_{j=0}^{i-1} \frac{(i-1)!(i+k-j-2)!x^j}{j!(i-1-j)!(\frac{1}{\alpha} + \frac{1}{\gamma})^{i+k-j-1}}, & x \geq 0 \\ e^{(\frac{1}{\alpha} + \frac{1}{\gamma})x} \sum_{j=0}^{k-1} \frac{(k-1)!(-x)^j}{j!(k-1-j)!} \frac{(k+i-2-j)!}{(\frac{1}{\alpha} + \frac{1}{\gamma})^{k+i-1-j}}, & x < 0 \end{cases}$$

we derive that

$$f_Z(x) = \begin{cases} \sum_{j=1}^{m_2} \eta_j f_{\mathcal{E}_{j,1/\gamma}}(x), & x \geq 0 \\ \sum_{j=1}^{m_1} \zeta_j f_{\mathcal{E}_{j,1/\alpha}}(-x), & x < 0 \end{cases}$$

where the quantities η_j, ζ_j appearing above are given by

$$\eta_j := \sum_{i=j}^{m_2} \sum_{k=1}^{m_1} h_i d_k \binom{i-j+k-1}{k-1} \frac{\alpha^{-k} \gamma^{j-i}}{(\frac{1}{\alpha} + \frac{1}{\gamma})^{k+i-j}} \quad \text{and} \quad \zeta_j := \sum_{i=1}^{m_2} \sum_{k=j}^{m_1} h_i d_k \binom{k+i-1-j}{i-1} \frac{\gamma^{-i} \alpha^{-k+j}}{(\frac{1}{\alpha} + \frac{1}{\gamma})^{k+i-j}},$$

with $\sum_{j=1}^{m_1} \zeta_j + \sum_{j=1}^{m_2} \eta_j = 1$. From the above form of f_Z it is evident that the random walk $S_k = u + \sum_{i=1}^k Z_i$, $k = 1, 2, \dots$ goes up or down at each step with probability $\sum_{j=1}^{m_2} \eta_j$ and $\sum_{j=1}^{m_1} \zeta_j$ respectively. The step up is a mixture of Erlang distributions with scale parameter $1/\gamma$ while the step down is a mixture of Erlang distributions with scale parameter $1/\alpha$. The moment generating function (m.g.f.) of the distribution of Z_i is given by

$$E(e^{wZ}) = E(e^{wcT_i})E(e^{-wX_i}) = \sum_{i=1}^{m_2} \frac{h_i}{(1-\gamma w)^i} \sum_{j=1}^{m_1} \frac{d_j}{(1+\alpha w)^j}, \quad -1/\alpha < w < 1/\gamma.$$

Denote by $w_1, w_2, \dots, w_m \in \mathbb{C}$ the m roots of the equation $E(e^{wZ}) = 1$ (excluding the root $w = 0$) or equivalently the roots of the polynomial equation (3.4). Since ν is a stopping time, we employ Wald's identity to get

$$E((E(e^{wZ}))^{-\nu} e^{wS'_\nu}) = 1, \quad \text{where} \quad S'_\nu := \sum_{i=1}^{\nu} Z_i = S_\nu - u,$$

and therefore, $E(e^{w_j S'_\nu}) = 1$, $j = 1, 2, \dots, m$. It follows that, for $j = 1, 2, \dots, m$,

$$1 = E(e^{w_j S'_\nu}) = E\left(e^{w_j S'_\nu} | S_\nu > b\right) P(S_\nu > b) + E\left(e^{w_j S'_\nu} | S_\nu < 0\right) (1 - P(S_\nu > b)). \quad (3.5)$$

The undershoot $-S_\nu$, given that $S_\nu < 0$, must have occurred after a downward step of the random walk $S_k, k = 1, 2, \dots$. Such a step can be considered as a sum of, at most m_1 , exponentially distributed r.v.'s. Invoking the memoryless property, the undershoot must also be a sum of at most m_1 exponentially distributed r.v.'s. In other words, the undershoot follows a mixture of Erlangs with c.d.f. $\sum_{i=1}^{m_1} \pi_i F_{\mathcal{E}_{i,1/\alpha}}(x), x \geq 0$, for some (unknown) non-negative weights $\pi_1, \pi_2, \dots, \pi_{m_1}$ such that $\sum_{i=1}^{m_1} \pi_i = 1$. This assertion could also be deduced by an argument similar to that in the proof of Lemma 7. Hence,

$$E\left(e^{w S'_\nu} | S_\nu < 0\right) = e^{-wu} E\left(e^{-w(-S_\nu)} | S_\nu < 0\right) = e^{-wu} \sum_{i=1}^{m_1} \frac{\pi_i}{(\alpha w + 1)^i}. \quad (3.6)$$

Analogously, the overshoot $S_\nu - b$, given that $S_\nu > b$, must have occurred after an upward step of the random walk $S_k, k = 1, 2, \dots$, and by the same reasoning as above, the overshoot follows a mixture of Erlangs with c.d.f. $\sum_{i=1}^{m_2} \kappa_i F_{\mathcal{E}_{i,1/\gamma}}(x), x \geq 0$, for some (unknown) non-negative weights $\kappa_1, \kappa_2, \dots, \kappa_{m_2}$ such that $\sum_{i=1}^{m_2} \kappa_i = 1$. Hence

$$E\left(e^{w S'_\nu} | S_\nu > b\right) = e^{w(b-u)} E\left(e^{w(S_\nu-b)} | S_\nu > b\right) = e^{w(b-u)} \sum_{i=1}^{m_2} \kappa_i \frac{1}{(1 - \gamma w)^i}. \quad (3.7)$$

Therefore, from (3.5), (3.6) and (3.7) above, we get the following system of m equations

$$1 = E(e^{w_j S'_\nu}) = e^{w_j(b-u)} \sum_{i=1}^{m_2} \frac{\kappa_i P(S_\nu > b)}{(1 - \gamma w_j)^i} + e^{-w_j u} \sum_{i=1}^{m_1} \frac{\pi_i (1 - P(S_\nu > b))}{(\alpha w_j + 1)^i}, \quad j = 1, 2, \dots, m$$

with m unknown quantities: $\pi_1, \dots, \pi_{m_1-1}, \kappa_1, \dots, \kappa_{m_2-1}$ and $P(S_\nu > b)$. If we set finally

$$a := \frac{P(S_\nu > b)}{1 - P(S_\nu > b)} \quad (\text{i.e. } P(S_\nu < 0) = 1 - P(S_\nu > b) = \frac{1}{1+a})$$

and replace π_{m_1} by $1 - \sum_{i=1}^{m_1-1} \pi_i$, and κ_{m_2} by $1 - \sum_{i=1}^{m_2-1} \kappa_i$, we equivalently get the system of m linear equations

$$\begin{aligned} 1 - \frac{e^{-w_j u}}{(\alpha w_j + 1)^{m_1}} &= \left(\frac{e^{w_j(b-u)}}{(1 - \gamma w_j)^{m_2}} - 1 \right) a + \sum_{i=1}^{m_2-1} \kappa_i a \left(\frac{e^{w_j(b-u)}}{(1 - \gamma w_j)^i} - \frac{e^{w_j(b-u)}}{(1 - \gamma w_j)^{m_2}} \right) \\ &\quad + \sum_{i=1}^{m_1-1} \pi_i \left(\frac{e^{-w_j u}}{(\alpha w_j + 1)^i} - \frac{e^{-w_j u}}{(\alpha w_j + 1)^{m_1}} \right), \quad j = 1, 2, \dots, m \end{aligned}$$

which can be solved w.r.t. $\pi_1, \dots, \pi_{m_1-1}, \kappa'_1, \dots, \kappa'_{m_2-1}, a$ ($\kappa'_i = a\kappa_i$) by inverting the matrix $\mathbf{B}$ as described in the theorem. $\square$

Note that results similar to Theorem 8 can be derived using the methodology described in Jacobsen (2011). In that paper the author uses martingale techniques to derive the joint generating function of the distributions of the overshoot/undershoot and the exit time of a random walk, offering explicit results for a class of jump distributions containing mixtures of exponentials. In Theorem 8, we use a different approach (based on Wald's identity) and we obtain the distribution function of the overshoot/undershoot in the context of the Sparre-Andersen model with mixed Erlang claim sizes and interarrival times.

Next, exploiting Theorem 8 we can express $E(S_\nu)$ and hence the mean values of R_ν (exit time) and ν (number of claims until exit time) as follows. To simplify the notation, we assume that T, X are generic r.v.'s with the same distributions as T_i and X_i respectively.

Proposition 9. *If the c.d.f.'s F, K of X_i 's and T_i 's are given by (3.3) then*

$$E(R_\nu) = \frac{E(S_\nu) - u}{cE(T) - E(X)} E(T) \quad (3.8)$$

where

$$E(T) = \beta^{-1} \sum_{i=1}^{m_2} i h_i, \quad E(X) = \alpha \sum_{i=1}^{m_1} i d_i, \quad E(S_\nu) = \frac{a}{1+a} \left(b + \frac{1}{\beta} \sum_{i=1}^{m_2} i \kappa_i \right) - \frac{\alpha}{1+a} \sum_{i=1}^{m_1} i \pi_i$$

and $a, \kappa_1, \dots, \kappa_{m_2}, \pi_1, \dots, \pi_{m_1}$ are given in Theorem 8.

Proof. Since ν is a stopping time with respect to the filtration $\mathcal{F}_i = \sigma((X_j, T_j), j = 1, 2, \dots, i)$, for $i = 1, 2, \dots$, we get from Wald's equation that

$$E(R_\nu) = E\left(\sum_{i=1}^{\nu} T_i\right) = E(\nu)E(T) \quad \text{and} \quad E(S_\nu) = E\left(u + \sum_{i=1}^{\nu} Z_i\right) = u + E(\nu)E(Z)$$

where $Z_i = cT_i - X_i$. Relation (3.8) follows by combining the above two relations. The expressions for $E(T)$ and $E(X)$ are straightforward from (3.3). Finally, the expectation of S_ν can be derived using Theorem 8 as follows

$$\begin{aligned} E(S_\nu) &= E(S_\nu | S_\nu > b) P(S_\nu > b) + E(S_\nu | S_\nu < 0) P(S_\nu < 0) \\ &= \left(b + \sum_{i=1}^{m_2} \kappa_i \frac{i}{\beta} \right) (1 - \psi_b(u)) - \sum_{i=1}^{m_1} i \pi_i \alpha \cdot \psi_b(u). \end{aligned}$$

□

From the above proof it is also evident that the mean number of claims until the process $S_k, k = 1, 2, \dots$ falls below 0 or exceeds b is

$$E(\nu) = \frac{E(R_\nu)}{E(T)} = \frac{E(S_\nu) - u}{cE(T) - E(X)}.$$

3.2 Mixed Erlang claims in the classical model

The above results concerning the renewal model are also valid in the classical risk model where the interarrival times between claims are exponentially distributed with mean $1/\beta = 1/\lambda$ ($m_2 = 1, h_1 = 1$). Moreover, in this case it can also be proved via Rouché's Theorem that, if $\rho := \frac{c}{\alpha\lambda} (\sum_{i=1}^m i d_i)^{-1} > 1$, the roots $w_1, w_2, \dots, w_m$ of the polynomial (3.4) lie within the open disk $D(-1/\alpha, (1-\varepsilon)/\alpha)$ of $\mathbb{C}$ centered at $-1/\alpha$ with radius $(1-\varepsilon)/\alpha$ for some $\varepsilon \in (0, 1)$. Hence $\operatorname{Re}(w_i) < -\varepsilon/\alpha < 0, i = 1, 2, \dots, m$, and thus

$$\left| e^{w_i b} \right| = \left| e^{\operatorname{Re}(w_i) b} e^{i \operatorname{Im}(w_i) b} \right| = e^{\operatorname{Re}(w_i) b} < e^{-\varepsilon b/\alpha} \xrightarrow{b \rightarrow \infty} 0, \quad i = 1, 2, \dots, m.$$

Therefore, if we assume that the upper barrier vanishes (i.e. $b \rightarrow \infty$) and $\rho > 1$, Theorem 8 is again valid, now with $B_{j1} = \frac{e^{w_j b}}{(1-\gamma w_j)^{m_2}} - e^{w_j u} \rightarrow -e^{w_j u}$, and $\tilde{\psi}_b(u) \rightarrow \psi(u)$, and $\tilde{F}_{de} \rightarrow F_{de,u}$.

Hence, in the classical model with mixed Erlang claims and no upper barrier, we can use Theorem 8 as described above to calculate the ruin probability and the distribution of the deficit at ruin. For this special case, a different method for the calculation of the same quantities was presented by Willmot and Woo (2007). Their approach relies mainly on the

inversion of the p.g.f. of a specific compound geometric distribution (cf. (3.10) in their paper), while our approach is based on the determination of the roots of a polynomial equation (cf. (3.4) above). Of course, both methods lead to the same numerical results. For example, when $\lambda = 5, c = 5, 1/\alpha = 10, u = 1$, and the mixing distribution is $d_i = i/55, i = 1, 2, \dots, m = 10$, (i.e. $\rho = 10/7 > 1$) both methods give

$$\psi(u) = 0.368875686047..., \text{ and } \pi_1 = 0.246819295870..., \quad \pi_2 = 0.203593452253... \quad \text{etc.}$$

The above numerical results were easily obtained through Wolfram Mathematica software, employing the functions `NSolve` and `SeriesCoefficient` for the two methods respectively (we verified that the numerical results from both methods agree up to at least 15 digits).

4 Examples and Computational Aspects

In this section we consider several specific cases in order to investigate the efficiency and the applicability of our results. Explicit results can be derived by assuming small values for m_1, m_2 , as it is shown in the first two examples. In more complicated cases our methods can still be implemented using computer algebra software. This is illustrated by the last two examples where we employ mixed Erlang distributions with as many as 100 weights. Taking into account that the mixed Erlang class is dense in the class of all non-negative distributions (see Remark 1), it is understood that, in practice, we can derive approximate results assuming almost any distribution for the claim sizes and interarrival times.

Example 3. At first we consider the simplest case when the interarrival times and the claim sizes are exponentially distributed with parameters $\lambda (= \beta)$ and $1/\mu (= 1/\alpha)$ respectively and put again $\gamma = c/\lambda$. In this case $m_1 = m_2 = 1$. We denote by $w_1 = 1/\gamma - 1/\mu = -R$ the $m = 1$ non-zero root of the polynomial equation (3.4). The $m \times m$ matrix $\mathbf{B}$ has only one element, i.e. $\mathbf{B} = (\frac{\mu}{\gamma} e^{-R(b-u)} - 1)$ and $\mathbf{a} = (1 - \frac{\gamma}{\mu} e^{Ru})$. The distributions $\tilde{F}_{de}$ and $\tilde{F}_{ov}$ are again trivially exponential distributions with parameters $1/\mu$ and $1/\gamma$ respectively, and

$$\pi = (a) = \mathbf{B}^{-1} \mathbf{a} = \left(\frac{1 - \frac{\gamma}{\mu} e^{Ru}}{\frac{\mu}{\gamma} e^{-R(b-u)} - 1} \right).$$

From Theorem 8 and (2.2) we get that

$$\tilde{\psi}_b(u) = P(S_\nu < 0) = \frac{1}{1+a} = \frac{e^{-Ru} - \frac{1}{\theta+1} e^{-Rb}}{1 + \theta - \frac{1}{\theta+1} e^{-Rb}}, \quad \psi_b(u) = \frac{e^{-Ru} - e^{-Rb}}{1 + \theta - e^{-Rb}},$$

where $\theta = \gamma/\mu - 1$ and $R = \theta/(\mu(1+\theta)) = \frac{\gamma-\mu}{\mu\gamma}$ is the adjustment coefficient. From (3.8) we also derive that

$$E(R_\nu) = \frac{\frac{a}{1+a} (b + \frac{1}{\lambda}) - \frac{\mu}{1+a} - u}{c - \mu\lambda},$$

and, for $x \geq 0$,

$$\begin{aligned} \tilde{J}_u(x) &= P(\tilde{M} \leq x | S_\nu < 0) = \int_0^x (1 - \tilde{\psi}_x(x-y)) d\tilde{F}_{de}(y) = 1 - \frac{(\theta+1)^2 - 1}{(\theta+1)^2 e^{Rx} - 1}, \\ J_u(x) &= P(M \leq x | \tau(u) < \infty) = \int_0^x (1 - \psi_x(x-y)) dF_{de,u}(y) = 1 - \frac{\theta}{(\theta+1) e^{Rx} - 1}. \end{aligned}$$

Note that $\tilde{J}_u, J_u$ are proper distributions only when $\gamma > \mu$ ($\Leftrightarrow \theta > 0$).

Example 4. Consider again the classical model and assume that $F(x) = d_1 F_{\mathcal{E}_{1,1/\alpha}}(x) + d_2 F_{\mathcal{E}_{2,1/\alpha}}(x)$, $x \geq 0$, $d_1 + d_2 = 1$, $d_2 > 0$ and $K(x) = F_{\mathcal{E}_{1,\lambda}}(x)$ ($\beta = \lambda$, $m_1 = 2$, $m_2 = 1$). The $m = 2$ non-zero roots w_1, w_2 of the polynomial equation 3.4 are

$$1/(2\gamma) - 1/\alpha \pm \sqrt{1/(4\gamma^2) + (1 - d_1)/(\gamma\alpha)}.$$

Since $(a, \pi_1)^T = \mathbf{B}^{-1}\mathbf{a}$ we get that

$$\begin{aligned} a &= \frac{\alpha e^{-u(w_1+w_2)}((w_1-w_2)+e^{uw_1}w_2(\alpha w_1+1)^2-e^{uw_2}w_1(\alpha w_2+1)^2)}{(\alpha w_1+1)^2(\alpha w_2+1)^2 \cdot \text{Det}(B)}, \\ \pi_1 &= \frac{(1 + \frac{e^{(b-u)w_2}}{\gamma w_2 - 1})(1 - \frac{e^{-uw_1}}{(\alpha w_1+1)^2}) - (1 - \frac{e^{(b-u)w_1}}{1-\gamma w_1})(1 - \frac{e^{-uw_2}}{(\alpha w_2+1)^2})}{\frac{\alpha e^{-uw_1}w_1}{(\alpha w_1+1)^2} + \frac{\alpha e^{bw_2-u(w_1+w_2)}w_1}{(\gamma w_2-1)(\alpha w_1+1)^2} - \frac{\alpha e^{-uw_2}w_2}{(\alpha w_2+1)^2} - \frac{\alpha e^{bw_1-u(w_1+w_2)}w_2}{(\gamma w_1-1)(\alpha w_2+1)^2}} \end{aligned}$$

($\pi_2 = 1 - \pi_1$). Thus $\tilde{\psi}_b(u) = (1 + a)^{-1}$ and $\tilde{F}_{de}(x) = \pi_1 F_{\mathcal{E}_{1,1/\alpha}}(x) + (1 - \pi_1) F_{\mathcal{E}_{2,1/\alpha}}(x)$, $x \geq 0$. Moreover, from (3.8) the mean exit time is

$$E(R_\nu) = \frac{\frac{a}{1+a} (b + \frac{1}{\lambda}) - \frac{\alpha}{1+a} (2 - \pi_1) - u}{c\frac{1}{\lambda} - \alpha(2 - d_1)} \cdot \frac{1}{\lambda}.$$

For larger values of m_1 and m_2 we can still apply Theorem 8. However, the complexity of the required algebraic manipulations may increase to a level that renders the whole procedure computationally very demanding, even for advanced computer algebra systems such as Mathematica or Maple. The task though is much easier when we assume specific values for the parameters $u, b, \alpha, \gamma = c/\beta$, $d_j, j = 1, 2, \dots, m_1$, $h_i, i = 1, 2, \dots, m_2$. Such cases are presented in the following Examples 5 and 6.

Example 5. Assume that the distribution of claims is mixed Erlang with parameters $1/\alpha = 2$, $m_1 = 2$, $d_1 = \frac{1}{5}$, $d_2 = \frac{4}{5}$ and the distribution of interarrival times is also a mixed Erlang with parameters $\beta = 3$, $m_2 = 3$, $h_1 = 0$, $h_2 = \frac{1}{3}$, $h_3 = \frac{2}{3}$.

Assume also that the rate of premium income flow is $c = 1$, the initial reserve is $u = 2$ and the upper barrier is $b = 6$. Three simulated paths of the insurer's surplus process $\{U(t), t \geq 0\}$ are shown in Figure 3. Two of these three paths hit the lower barrier 0 while the third path crosses the upper barrier $b = 6$ before ruin occurs.

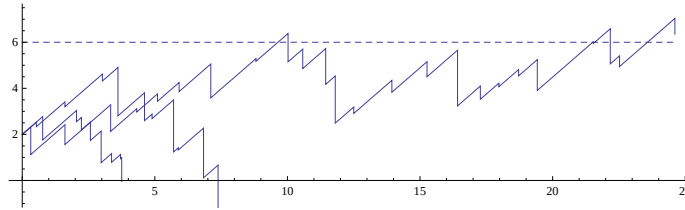


Fig. 3. Paths of the surplus process.

We shall calculate $\tilde{\psi}_b(u)$ using Theorem 8. The $m = 4$ non-zero roots of the polynomial equation are (evaluated numerically),

$$\begin{aligned} w_1 &\approx -2.736205696197294 & w_3 &\approx 3.85435991869825 - 1.0487511263573713 \mathbf{i} \\ w_2 &\approx 0.0274858588007930 & w_4 &\approx 3.85435991869825 + 1.0487511263573713 \mathbf{i}. \end{aligned}$$

After some algebra, we find using Wolfram Mathematica that

$$\pi = \mathbf{B}^{-1}\mathbf{a} \approx (0.5271523991, 0.2973415430, 0.169357717, 0.7303037761)^T$$

and therefore,

$$\tilde{\psi}_b(u) = P(S_\nu < 0) = (1 + a)^{-1} \approx 0.654813495.$$

The distribution of the deficit at ruin $\tilde{F}_{de}$ is mixed Erlang with parameters $\alpha = 1/2, \pi_1 \approx 0.73030377, \pi_2 \approx 0.26969622$ and the distribution $\tilde{F}_{ov}$ of the overshoot is mixed Erlang with parameters $\gamma = 1/3$ and $\kappa_1 \approx 0.564052337, \kappa_2 \approx 0.32126898, \kappa_3 \approx 0.114678675$. Moreover, from (3.8) we get that the mean time to exit is $E(R_\nu) \approx 13.2935722$.

The shapes of the densities for the distributions $\tilde{F}_{ov}, \tilde{F}_{de}$ are shown in the following graphs (solid line). For comparison we also include the histograms of the same distributions generated by simulating 10^5 paths of the process S .

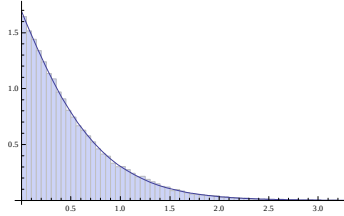


Fig. 4. Density of $\tilde{F}_{ov}$

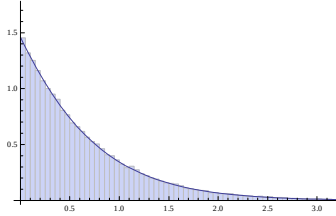


Fig. 5. Density of $\tilde{F}_{de}$

From the same 10^5 simulated paths we also estimate $\tilde{\psi}_b(u) = \tilde{\psi}_6(2) \approx 0.6572$ and $E(R_\nu) \approx 13.2963$, which agree with the calculated values from Theorem 8.

Example 6. It is remarkable that the numeric computation of ruin related quantities via Theorem 8 can be accomplished even for moderately large values of m_1, m_2 . As a last example we consider that $1/\alpha = 20, m_1 = 100, d_i = (i - \frac{m_1}{2})^2/83350$ and $\beta = 20, m_2 = 100, h_i = i/5050$. The shapes of the p.d.f.'s of X and T can be seen in the following graphs:

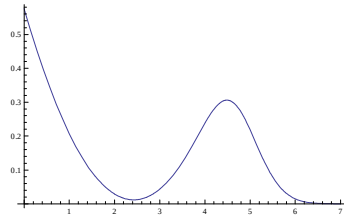


Fig.6. Density of claims

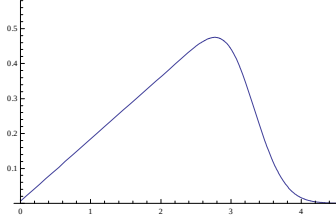


Fig.7. Density of interarrival times

We also assume that $c = 1, b = 10$ and $u = 2$. The polynomial in (3.4) has $m = 199$ non-zero roots (and one zero root) that are shown in the following figure.

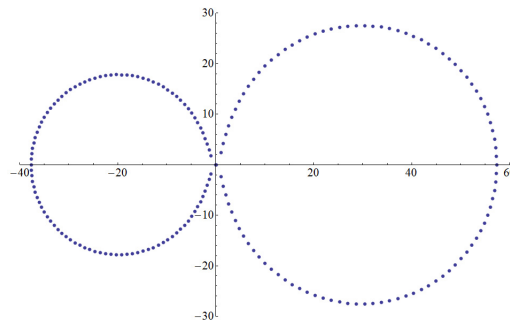


Fig.8. The $m + 1$ roots of the polynomial in the complex plane

By inverting the $m \times m$ matrix $\mathbf{B}$ we get, according to Theorem 8 and (3.8), that

$$\tilde{\psi}_b(u) \approx 0.884867445, \quad E(R_\nu) \approx 11.7185249.$$

The Monte Carlo estimates are $\tilde{\psi}_b(u) \approx 0.8834$, $E(R_\nu) \approx 11.663$ (via 10^5 iterations). Finally, the shapes of the densities for $\tilde{F}_{ov}$, $\tilde{F}_{de}$ derived from Theorem 8 are shown in the following graphs (solid line). Again, we also include the respective histograms generated by simulating 10^5 paths of the process S .

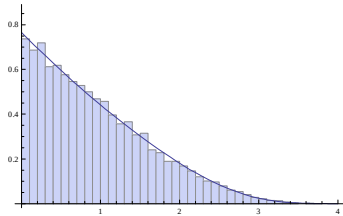


Fig. 9. Density of $\tilde{F}_{ov}$

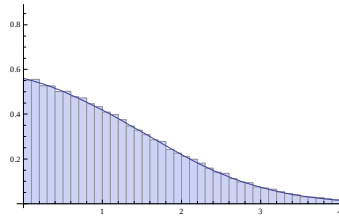


Fig. 10. Density of $\tilde{F}_{de}$

All computations were performed using Wolfram Mathematica (requiring 50 digits precision).

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