Supplementary material to the paper

Exact prediction intervals for order statistics from the Laplace distribution based on the MLEs

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Below we explicitly derive the conditional distributions of the pivotal quantities which in the paper are presented in Tables 1, 2 and 3. Moreover, we provide the proofs of the results stated in paper's Appendix, namely, Propositions 3,4,5 and 6.

3 The exact distributions of $T_1(n,r,s,k)$ and $T_2(n,r,s,k)$

3.1 The exact distribution of $T_1(n,r,s,k)$

Case $\max(r,s) < m$

Consider first the case $\max(r,s) < m$. As in Iliopoulos and Balakrishnan (2011), it is convenient to write $T_1/(n-r-s)$ in different forms, depending on which value d of D we condition on. In order not to increase the paper's length considerably, we derive the conditional distribution of T_1 given $D=d$ explicitly only for a particular range of d's. For the remaining cases we give just the result without many details.

$r < d \leqslant m-1$: Conditional on $D=d$ we have

$$T_1/(n-r-s) = \frac{X_{n-s+k:n} - X_{n-s:n}}{-rX_{r+1:n} - \sum_{i=r+1}^{d} X_{i:n} - \sum_{i=d+1}^{[n/2]} X_{i:n} + \sum_{m+1}^{n-s} X_{i:n} + sX_{n-s:n}}$$

(by making the convention $\sum_{i=k}^{\ell} \equiv 0$ when $k > \ell$)

$$\stackrel{\mathrm{d}}{=} \frac{R_{n-s+k-d:n-d} - R_{n-s-d:n-d}}{rL_{d-r:d} + \sum_{i=1}^{d-r} L_{i:n} - \sum_{i=1}^{[n/2]-d} R_{i:n-d} + \sum_{m+1-d}^{n-s-d} R_{i:n-d} + sR_{n-s-d:n-d}}$$

(where $R_1,\ldots,R_{n-d},L_1,\ldots,L_d \stackrel{\text{iid}}{\sim} \mathcal{E}(1)$; see Section 3.1 of the paper)

$$= \frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

(where $\tilde{R}_{i:k} = R_{i:k} - R_{i-1:k}$, $\tilde{L}_{i:k} = L_{i:k} - L_{i-1:k}$; see Section 3.1 of the paper)

$$\stackrel{d}{=} \frac{\sum_{i=n-s-d+1}^{n-s-d+k} \frac{1}{n-d-i+1} R_i}{\sum_{i=1}^{d-r} L_i + \sum_{i=1}^{m-d} \frac{d+i-1}{n-d-i+1} R_i + \sum_{i=m-d+1}^{n-s-d} R_i},$$

(by using the properties of normalized spacings of exponential order statistics).

It can be verified that the last fraction has cdf $F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a)$, see Proposition 3(a), with $p = k$, $q = m - d$, $\boldsymbol{\eta} = (s - k + 1, \ldots, s)$, $\boldsymbol{\theta} = ((n - d)/d, \ldots, (n - m + 1)/(m - 1))$, and $a = n - m - s - r + d$.

$\underline{0 \leqslant d \leqslant r}$: In this case the conditional distribution of $T_1/(n - r - s)$, given $D = d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{m-d} (d + i - 1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-s-d} (n - d - i + 1)\tilde{R}_{i:n-d}}.$$

By normalizing the spacings $\tilde{R}$ accordingly, we conclude that this fraction has also cdf $F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a)$, with $p = k$, $q = m - r - 1$, $\boldsymbol{\eta} = (s - k + 1, \ldots, s)$, $\boldsymbol{\theta} = ((n - r - 1)/(r + 1), \ldots, (n - m + 1)/(m - 1))$, and $a = n - m - s$.

$\underline{d = m \neq n - s}$: Here the conditional distribution of $T_1/(n - r - s)$, given $D = d$, is the same as that of

$$\frac{\sum_{i=n-m-s+1}^{n-m-s+k} \tilde{R}_{i:n-m}}{[n/2]\tilde{L}_{1:m} + \sum_{i=1}^{m-r} (m - i + 1)\tilde{L}_{i:m} + \sum_{i=1}^{n-m-s} (n - m - i + 1)\tilde{R}_{i:n-m}}.$$

The distribution of this fraction depends on whether $n = 2m - 1$ or $n = 2m$. In the first case it has cdf $F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a)$ with $p = k$, $q = 1$, $\boldsymbol{\eta} = (s - k + 1, \ldots, s)$, $\boldsymbol{\theta} = (m/(m - 1))$, and $a = n - s - r - 1$, while in the second it has cdf $F_1(y; \boldsymbol{\eta}, \theta = \varnothing, a)$, see Proposition 3(c), with p, $\boldsymbol{\eta}$ as before and $a = n - s - r$.

$\underline{m + 1 \leqslant d < n - s}$: In this case the conditional distribution of $T_1/(n - r - s)$, given $D = d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-[n/2]} (n - d + i - 1)\tilde{L}_{i:d} + \sum_{i=d-[n/2]+1}^{d-r} (d - i + 1)\tilde{L}_{i:d} + \sum_{i=1}^{n-d-s} (n - d - i + 1)\tilde{R}_{i:n-d}}.$$

The cdf of this fraction is again $F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a)$ with $p = k$, $q = d - [n/2]$, $\boldsymbol{\eta} = (s - k + 1, \ldots, s)$, $\boldsymbol{\theta} = (d/(n - d), \ldots, (n - m + 1)/(m - 1))$, and $a = 2n - m - r - s - d$.

$\underline{n - s \leqslant d < n - s + k}$: Conditional on $D = d$ for $d \geqslant n - s$ the numerator of the fraction takes a different form because this implies that all observed data are smaller than zero (the median). The conditional distribution of $T_1/(n - r - s)$, given $D = d$, is the same as that of

$$\frac{L_{d-n+s+1:d} + R_{n-s-d+k:n-d}}{-sL_{d-n+s+1:d} - \sum_{i=d-n+s+1}^{d-m} L_{i:d} + \sum_{i=d-[n/2]+1}^{d-r} L_{i:d} + rL_{d-r:d}}$$

$$= \frac{\sum_{i=1}^{d-n+s+1} \tilde{L}_{i:d} + \sum_{i=1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=d-n+s+2}^{d-[n/2]} (n-d+i-1)\tilde{L}_{i:d} + \sum_{i=d-[n/2]+1}^{d-r} (d-i+1)\tilde{L}_{i:d}}.$$

This has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k+1$, $q=m-s-1$, $\boldsymbol{\eta}=(s-k+1,\dots,n-d,n-s,\dots,d)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\dots,(n-m+1)/(m-1))$, and $a=n-m-r$. It is worth noting here that the η_i's are distinct since $s \leqslant m-1 < n/2$ which implies $s-k+1<n-s$.

$\underline{n-s+k \leqslant d \leqslant n}$: Finally, in this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=d-n+s-k+2}^{d-n+s+1} \tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-[n/2]} (n-d+i-1)\tilde{L}_{i:d} + \sum_{i=d-[n/2]+1}^{d-r} (d-i+1)\tilde{L}_{i:d}},$$

whose distribution is again $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k$, $q=m-s-1$, $\boldsymbol{\eta}=(n-s,\dots,n-s+k-1)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\dots,(n-m+1)/(m-1))$, and $a=n-m-r$.

Case $s \geqslant m$

$\underline{0 \leqslant d \leqslant r}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{n-d-s} (d+i-1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a=0)$ given in Proposition 3(b), with $p=k$, $q=n-r-s-1$, $\boldsymbol{\eta}=(s-k+1,\dots,s)$, and $\boldsymbol{\theta}=((n-r-1)/(r+1),\dots,(s+1)/(n-s-1))$.

$\underline{r<d<n-s}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r} (d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{n-s-d} (d+i-1)\tilde{R}_{i:n-d}}.$$

This fraction has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k$, $q=n-s-d$, $\boldsymbol{\eta}=(s-k+1,\dots,s)$, $\boldsymbol{\theta}=((n-d)/d,\dots,(s+1)/(n-s-1)$, and $a=d-r$.

$\underline{n-s \leqslant d < n-s+k}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=1}^{d-n+s+1} \tilde{L}_{i:d} + \sum_{i=1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=d-n+s+2}^{d-r} (d-i+1)\tilde{L}_{i:d}}.$$

Here there are two possibilities depending on whether the entries of the $(k+1)$-dimensional vector $\boldsymbol{\delta}=(s-k+1,\dots,n-d,n-s,\dots,d)$ are distinct or not. If this is the case then the fraction has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$ with $p=k+1$, $\boldsymbol{\eta}=\boldsymbol{\delta}$, and $a=n-r-s-1$. If on the other hand there are $q \geqslant 1$ ties (pairs) among the entries of $\boldsymbol{\delta}$, then split it in two parts and call $\boldsymbol{\eta}$ the $(k+1-2q)$-dimensional vector of the untied entries and $\boldsymbol{\theta}$ the q-dimensional vector of the tied entries (each taken once). Then, the cdf of the fraction is $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda}=\varnothing,\boldsymbol{\mu}=\varnothing,a)$, given in Proposition 5(d), where $p=k+1-2q$ and a as before.

$\underline{n-s+k \leqslant d \leqslant n}$: Finally, in this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=d-n+s-k+2}^{d-n+s+1}\tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-r}(d-i+1)\tilde{L}_{i:d}}$$

which has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$ with $p=k$, $\boldsymbol{\eta}=(n-s,\dots,n-s+k-1)$ and $a=n-r-s-1$.

Case $r \geqslant m$

$\underline{0 \leqslant d \leqslant r}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k}\tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}},$$

whose cdf is $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$ with $p=k$, $\boldsymbol{\eta}=(s-k+1,\dots,s)$, and $a=n-r-s-1$.

$\underline{r < d < n-s}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=n-s-d+1}^{n-s-d+k}\tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k$, $q=d-r$, $\boldsymbol{\eta}=(s,\dots,s-k+1)$, $\boldsymbol{\theta}=(d/(n-d),\dots,(r+1)//(n-r-1))$, and $a=n-s-d$.

$\underline{n-s \leqslant d < n-s+k}$: In this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=1}^{d-n+s+1}\tilde{L}_{i:d}+\sum_{i=1}^{n-s-d+k}\tilde{R}_{i:n-d}}{\sum_{i=d-n+s+2}^{d-r}(n-d+i-1)\tilde{L}_{i:d}}.$$

This has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a=0)$ with $p=k+1$, $q=n-r-s-1$, $\boldsymbol{\eta}=(s-k+1,\dots,n-d,n-s,\dots,d)$, and $\boldsymbol{\theta}=((n-s-1)/(s+1),\dots,(r+1)/(n-r-1))$. Note that here the η_i's are distinct.

$\underline{n-s+k \leqslant d \leqslant n}$: Finally, in this case the conditional distribution of $T_1/(n-r-s)$, given $D=d$, is the same as that of

$$\frac{\sum_{i=d-n+s-k+2}^{d-n+s+1}\tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-r}(n-d+i-1)\tilde{L}_{i:d}}$$

which again has cdf $F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k$, $q=n-r-s-1$, $\boldsymbol{\eta}=(n-s,\dots,n-s+k-1)$, and $\boldsymbol{\theta}=((n-s-1)/(s+1),\dots,(r+1)/(n-r-1))$.

4 The exact distribution of $T_3(n, r, s, n', k)$

Let us now proceed to determine the exact distribution of $T_3(n, r, s, n', k)$ in (3) of the paper. This distribution is also free of the parameters μ and σ and thus we may again take without loss of generality $\mu = 0$ and $\sigma = 1$. Moreover, in addition to the random variable D defined in the previous section, we introduce the corresponding random variable related to the future sample. So, let $D' = \#\{Y's \leqslant 0\}$. Obviously, D' can be taken independent of D. Below we will determine the conditional distribution of T_3 given $D = d$ *and* $D' = d'$ for all pairs $(d, d') \in \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n'\}$. If we set $P(T_3 \leqslant t | D = d, D' = d') = F^{(3)}(t|d, d')$, then we have

$$P(T_3 \leqslant t) \equiv F^{(3)}(t) = \frac{1}{2^{n+n'}} \sum_{d=0}^{n} \sum_{d'=0}^{n'} \binom{n}{d} \binom{n'}{d'} F^{(3)}(t|d, d'), \quad t \in \mathbb{R}. \tag{1}$$

In order to present the conditional distributions of T_3 we will further need two independent sequences of iid standard exponential random variables which will be denoted by $L'_1, L'_2, \ldots$ and $R'_1, R'_2, \ldots$. These sequences will be related with the second sample and so they will be considered independent of the L- and R-sequences introduced in the previous section. The corresponding spacings will be denoted by $\tilde{L}'$ and $\tilde{R}'$, respectively.

For further use, observe that conditional on $D' = d'$ (and $D = d$),

$$Y_{k:n'} \stackrel{\mathrm{d}}{=} \begin{cases} R'_{k-d':n'-d'}, & 0 \leqslant d' < k \\ -L'_{d'-k+1:d'}, & k \leqslant d' \leqslant n' \end{cases} = \begin{cases} \sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'}, & 0 \leqslant d' < k \\ -\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'}, & k \leqslant d' \leqslant n' \end{cases}$$
$$\stackrel{\mathrm{d}}{=} \begin{cases} \sum_{i=1}^{k-d'} (n' - d' - i + 1)^{-1} R_i, & 0 \leqslant d' < k \\ -\sum_{i=1}^{d'-k+1} (d' - i + 1)^{-1} L_i, & k \leqslant d' \leqslant n'. \end{cases}$$

Case $\max(r, s) < m$

<u>$0 \leqslant d \leqslant r$ and $0 \leqslant d' < k$</u>: When $n = 2m-1$, the conditional distribution of $T_3/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{R'_{k-d':n'-d'} - R_{m-d:n-d}}{-rR_{r-d+1:n-d} - \sum_{i=r-d+1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} R_{i:n-d} + sR_{n-d-s:n-d}}$$
$$= \frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{m-d} (d + i - 1) \tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} (n - d - i + 1) \tilde{R}_{i:n-d}}.$$

By normalizing these spacings accordingly, we find out that the cdf of the above fraction is $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$, see Proposition 4(a), with $p = k - d'$, $q = r - d + 1$, $h = m - r - 1$, $\boldsymbol{\eta} = (n' - k + 1, \ldots, n' - d')$, $\boldsymbol{\theta} = (n - r, \ldots, n - d)$, $\boldsymbol{\lambda} = (1/m, \ldots, 1/(n - r - 1))$,

$\boldsymbol{\mu} = ((m-1)/m, \ldots, (r+1)/(n-r-1))$, and $a = m-s-1$. On the other hand, when $n = 2m$ we have, conditional on $D = d$, $D' = d'$, $T_3/(n-r-s)$ to have the same distribution as

$$\frac{R'_{k-d':n'-d'} - \frac{1}{2}R_{m-d:n-d} - \frac{1}{2}R_{m-d+1:n-d}}{-rR_{r-d+1:n-d} - \sum_{i=r-d+1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} R_{i:n-d} + sR_{n-d-s:n-d}}$$
$$= \frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d} - \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=r-d+2}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has also cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, q, a, $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ as before and $h = m-r$, $\boldsymbol{\lambda} = (1/2m, 1/(m+1), \ldots, 1/(n-r-1))$ and $\boldsymbol{\mu} = (1, (m-1)/(m+1), \ldots, (r+1)/(n-r-1))$.

$0 \leqslant d \leqslant r$ and $k \leqslant d' \leqslant n'$: When $n = 2m-1$ the conditional distribution of $-T_3/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} + \sum_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$, see Proposition 5(a), with q the number of ties (pairs) in $\boldsymbol{\delta} = (k, \ldots, d', n-r, \ldots, n-d)$, $\boldsymbol{\theta}$ the vector of tied entries (each taken once), $\boldsymbol{\eta}$ the vector of untied entries, $p = d'-k+r-d+2-2q$, $h = m-r-1$, $\boldsymbol{\lambda} = (1/m, \ldots, 1/(n-r-1))$, $\boldsymbol{\mu} = ((m-1)/m, \ldots, (r+1)/(n-r-1))$ and $a = m-s-1$. When $n = 2m$, conditional on $D = d$, $D' = d'$, $-T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} + \sum_{i=1}^{m-d} \tilde{R}_{i:n-d} + \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=r-d+2}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has also cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, q, a, $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$ as before and $h = m-r$, $\boldsymbol{\lambda} = (1/2m, 1/(m+1), \ldots, 1/(n-r-1))$ and $\boldsymbol{\mu} = (1, (m-1)/(m+1), \ldots, (r+1)/(n-r-1))$.

$r+1 \leqslant d \leqslant m-1$ and $0 \leqslant d' < k$: When $n = 2m-1$, the conditional distribution of $T_3/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta} = \varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$, see Proposition 4(b), with $p = k-d'$, $h = m-d$, $\boldsymbol{\eta} = (n'-k+1, \ldots, n'-d')$, $\boldsymbol{\lambda} = (1/m, \ldots, 1/(n-d))$, $\boldsymbol{\mu} = ((m-1)/m, \ldots, d/(n-d))$, and $a = m-r-s+d-1$. When $n = 2m$, the conditional distribution of $T_3/(n-r-s)$ given $D = d$, $D' = d'$ is the same with that of

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d} - \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has also cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, a and $\boldsymbol{\eta}$ as before and $h=m-d+1$, $\boldsymbol{\lambda}=(1/2m,1/(m+1),\dots,1/(n-d))$, $\boldsymbol{\mu}=(1,(m-1)/(m+1),\dots,d/(n-d))$.

$r+1\leqslant d\leqslant m-1$ and $k\leqslant d'\leqslant n'$: When $n=2m-1$, the conditional distribution of $-T_3/(n-r-s)$, given $D=d$, $D'=d'$, is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}+\sum_{i=1}^{m-d}\tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d}+\sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This fraction has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$, see Proposition 5(b), with $p=d'+1-k$, $h=m-d$, $\boldsymbol{\eta}=(k,\dots,d')$, $\boldsymbol{\lambda}=(1/m,\dots,1/(n-d))$, $\boldsymbol{\mu}=((m-1)/m,\dots,d/(n-d))$, and $a=m-r-s+d-1$. When $n=2m$, the conditional distribution of $-T_3/(n-r-s)$ is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}+\sum_{i=1}^{m-d}\tilde{R}_{i:n-d}+\frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d}+\sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, a and $\boldsymbol{\eta}$ as before and $h=m-d+1$, $\boldsymbol{\lambda}=(1/2m,1/(m+1),\dots,1/(n-d))$, and $\boldsymbol{\mu}=(1,(m-1)/(m+1),\dots,d/(n-d))$.

$d=m\neq n-s$ and $0\leqslant d'<k$: For $n=2m-1$, conditional on $D=m$ and $D'=d'$, $T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'}\tilde{R}'_{i:n'-d'}+L_{1:m}}{(m-1)\tilde{L}_{1:m}+\sum_{i=1}^{m-r}(m-i+1)\tilde{L}_{i:m}+\sum_{i=1}^{n-m-s}(n-m-i+1)\tilde{R}_{i:n-m}}.$$

This has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d'$, $h=1$, $\boldsymbol{\eta}=(n'-k+1,\dots,n'-d')$, $\boldsymbol{\lambda}=(1/m)$, $\boldsymbol{\mu}=((m-1)/m)$, and $a=n-r-s-1$. When $n=2m$, conditional on $D=d$, $D=d'$, $T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'}\tilde{R}'_{i:n'-d'}-\frac{1}{2}R_{1:m}+\frac{1}{2}L_{1:m}}{m\tilde{L}_{1:m}+\sum_{i=1}^{m-r}(m-i+1)\tilde{L}_{i:m}+\sum_{i=1}^{n-m-s}(n-m-i+1)\tilde{R}_{i:n-m}}.$$

This has cdf $F_4(y;\boldsymbol{\eta},\boldsymbol{\lambda},a)$, see Proposition 6, with p, $\boldsymbol{\eta}$ as before, $\boldsymbol{\lambda}=(1/2,1/2)$, and $a=n-r-s-2$.

$d=m\neq n-s$ and $k\leqslant d'\leqslant n'$: When $n=2m-1$, $-T_3/(n-r-s)$ given $D=m$, $D'=d'$, has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}-L_{1:m}}{(m-1)\tilde{L}_{1:m}+\sum_{i=1}^{m-r}(m-i+1)\tilde{L}_{i:m}+\sum_{i=1}^{n-m-s}(n-m-i+1)\tilde{R}_{i:n-m}}.$$

This has cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k$, $h=1$, $\boldsymbol{\eta}=(k,\dots,d')$, $\boldsymbol{\lambda}=(1/m)$, $\boldsymbol{\mu}=((m-1)/m)$, and $a=n-r-s-1$. When $n=2m$, $-T_3/(n-r-s)$ given $D=d$, $D=d'$ has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}+\frac{1}{2}R_{1:m}-\frac{1}{2}L_{1:m}}{m\tilde{L}_{1:m}+\sum_{i=1}^{m-r}(m-i+1)\tilde{L}_{i:m}+\sum_{i=1}^{n-m-s}(n-m-i+1)\tilde{R}_{i:n-m}}.$$

This has cdf $F_4(y;\boldsymbol{\eta},\boldsymbol{\lambda},a)$ as well with p and $\boldsymbol{\eta}$ as before, $\boldsymbol{\lambda}=(1/2,1/2)$, and $a=n-r-s-2$.

$\underline{m+1\leqslant d<n-s \text{ and } 0\leqslant d'<k}$: When $n=2m-1$, conditional on $D=d$ and $D'=d'$, $T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'}\tilde{R}'_{i:n'-d'}+\sum_{i=1}^{d-m+1}\tilde{L}_{i:d}}{\sum_{i=1}^{d-m+1}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=d-m+2}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}},$$

whose distribution is $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d'$, $h=d-m+1$, $\boldsymbol{\eta}=(n'-k+1,\ldots,n'-d')$, $\boldsymbol{\lambda}=(1/m,\ldots,1/d)$, $\boldsymbol{\mu}=((m-1)/m,\ldots,(n-d)/d)$ and $a=n-r-s-d+m-1$. When $n=2m$, $T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'}\tilde{R}'_{i:n'-d'}+\sum_{i=1}^{d-m}\tilde{L}_{i:d}+\frac{1}{2}\tilde{L}_{d+1-m:d}}{\sum_{i=1}^{d-m}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=d-m+1}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, h and $\boldsymbol{\eta}$ as before and $\boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/d)$, $\boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,(n-d)/d)$, $a=n-r-s-d+m-1$.

$\underline{m+1\leqslant d<n-s \text{ and } k\leqslant d'\leqslant n'}$: When $n=2m-1$, conditional on $D=d$ and $D'=d'$, $-T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}-\sum_{i=1}^{d-m+1}\tilde{L}_{i:d}}{\sum_{i=1}^{d-m+1}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=d-m+2}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k$, $h=d-m+1$, $\boldsymbol{\eta}=(k,\ldots,d')$, $\boldsymbol{\lambda}=(1/m,\ldots,1/d)$, $\boldsymbol{\mu}=((m-1)/m,\ldots,(n-d)/d)$, and $a=n-r-s-d+m-1$. When $n=2m$, $-T_3/(n-r-s)$ given $D=d$, $D'=d'$, has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1}\tilde{L}'_{i:d'}-\sum_{i=1}^{d-m}\tilde{L}_{i:d}-\frac{1}{2}\tilde{L}_{d+1-m:d}}{\sum_{i=1}^{d-m}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=d-m+1}^{d-r}(d-i+1)\tilde{L}_{i:d}+\sum_{i=1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with p, h and $\boldsymbol{\eta}$ as before, $\boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/d)$, $\boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,(n-d)/d)$, and $a=n-r-s-d+m-1$.

$\underline{n-s\leqslant d\leqslant n \text{ and } 0\leqslant d'<k}$: When $n=2m-1$, conditional on $D=d$ and $D'=d'$, $T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'}\tilde{R}'_{i:n'-d'}+\sum_{i=1}^{d-m+1}\tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-m+1}(n-d+i-1)\tilde{L}_{i:d}+\sum_{i=d-m+2}^{d-r}(d-i+1)\tilde{L}_{i:d}}.$$

This has cdf $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with q the number of ties (pairs) in $\boldsymbol{\delta}=(n'-k+1,\ldots,n'-d',n-s,\ldots,d)$, $\boldsymbol{\theta}$ the vector of tied entries (each taken once), $\boldsymbol{\eta}$ the vector of untied entries, $p=k-d'+d-n+s+1-2q$, $h=m-s-1$, $\boldsymbol{\lambda}=(1/m,\ldots,1/(n-s-1))$,

$\boldsymbol{\mu} = ((m-1)/m, \ldots, (s+1)/(n-s-1))$, and $a = m-r-1$. When $n = 2m$, $T_3/(n-r-s)$ given $D = d$, $D' = d'$ has the same distribution as

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} + \sum_{i=1}^{d-m} \tilde{L}_{i:d} + \frac{1}{2m}\tilde{L}_{d+1-m:d}}{\sum_{i=d-n+s+2}^{d-m}(n-d+i-1)\tilde{L}_{i:d} + \sum_{i=d-m+1}^{d-r}(d-i+1)\tilde{L}_{i:d}}.$$

This fraction has cdf $F_3(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with p, q, $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ and a as before and $h = m-s$, $\boldsymbol{\lambda} = (1/2m, 1/(m+1), \ldots, 1/(n-s-1))$ $\boldsymbol{\mu} = (1, (m-1)/(m+1), \ldots, (s+1)/(n-s-1))$.

$n-s \leqslant d \leqslant n$ and $k \leqslant d' \leqslant n'$: When $n = 2m-1$, conditional on $D = d$ and $D' = d'$, $-T_3/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} - \sum_{i=1}^{d-m+1} \tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-m+1}(n-d+i-1)\tilde{L}_{i:d} + \sum_{i=d-m+2}^{d-r}(d-i+1)\tilde{L}_{i:d}}.$$

This fraction has cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with $p = d'+1-k$, $q = d-n+s+1$, $h = m-s-1$, $\boldsymbol{\eta} = (k, \ldots, d')$, $\boldsymbol{\theta} = (n-s, \ldots, d)$, $\boldsymbol{\lambda} = (1/m, \ldots, 1/(n-s-1))$, $\boldsymbol{\mu} = ((m-1)/m, \ldots, (s+1)/(n-s-1))$, and $a = m-r-1$. When $n = 2m$, $-T_3/(n-r-s)$ given $D = d$, $D' = d'$ has the same distribution as

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} - \sum_{i=1}^{d-m} \tilde{L}_{i:d} - \frac{1}{2}\tilde{L}_{d+1-m:d}}{\sum_{i=d-n+s+2}^{d-m}(n-d+i-1)\tilde{L}_{i:d} + \sum_{i=d-m+1}^{d-r}(d-i+1)\tilde{L}_{i:d}}.$$

This has cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with p, q, $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ and a as before and $h = m-s$, $\boldsymbol{\lambda} = (1/2m, 1/(m+1), \ldots, 1/(n-s-1))$, $\boldsymbol{\mu} = (1, (m-1)/(m+1), \ldots, (s+1)/(n-s-1))$.

Case $s \geqslant m$

Note that when $s \geqslant m$,

$$T_3(n, r, s, n', k) = \frac{Y_{k:n'} - \hat{\mu}_{\text{MLE}}}{\hat{\sigma}_{\text{MLE}}} = \frac{Y_{k:n'} - X_{n-s:n}}{\hat{\sigma}} - \log\left(\frac{n}{2(n-s)}\right) \tag{2}$$

and thus, we actually need to find the distribution of $T_3^* = (Y_{k:n'} - X_{n-s:n})/\hat{\sigma}_{\text{MLE}}$.

$0 \leqslant d \leqslant r$ and $0 \leqslant d' < k$: In this case, the conditional distribution of $T_3^*/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{n-s-d} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{n-d-s}(d+i-1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a = 0)$, see Proposition 4(c), with $p = k-d'$, $q = r-d+1$, $h = n-r-s-1$, $\boldsymbol{\eta} = (n'-k+1, \ldots, n'-d')$, $\boldsymbol{\theta} = (n-r, \ldots, n-d)$, $\boldsymbol{\lambda} = (1/(n-r-1), \ldots, 1/(s+1))$, and $\boldsymbol{\mu} = ((r+1)/(n-r-1), \ldots, (n-s-1)/(s+1))$.

$\underline{0 \leqslant d \leqslant r \text{ and } k \leqslant d' \leqslant n'}$: In this case, the conditional distribution of $-T_3^*/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} + \sum_{i=1}^{n-s-d} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{n-d-s}(d+i-1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_3(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a = 0)$ presented in Proposition 5(c), with q the number of ties (pairs) in $\boldsymbol{\delta} = (k, \ldots, d', n-r, \ldots, n-d)$, $\boldsymbol{\theta}$ the vector of tied entries (each taken once), $\boldsymbol{\eta}$ the vector of untied entries, $p = d' - k + r - d + 2 - 2q$ and h, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ as before.

$\underline{r+1 \leqslant d < n-s \text{ and } 0 \leqslant d' < k}$: Here, the conditional distribution of $T_3^*/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{n-s-d} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{n-s-d}(d+i-1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta} = \varnothing, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with $p = k-d'$, $h = n-s-d$, $\boldsymbol{\eta} = (n'-k+1, \ldots, n'-d')$, $\boldsymbol{\lambda} = (1/(s+1), \ldots, 1/(n-d))$, $\boldsymbol{\mu} = ((n-s-1)/(s+1), \ldots, d/(n-d))$, and $a = d - r$.

$\underline{r+1 \leqslant d < n-s \text{ and } k \leqslant d' \leqslant n'}$: In this case, the conditional distribution of $-T_3^*/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} + \sum_{i=1}^{n-s-d} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{n-s-d}(d+i-1)\tilde{R}_{i:n-d}}.$$

This has cdf $F_3(y; \boldsymbol{\eta}, \boldsymbol{\theta} = \varnothing, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with $p = d' - k + 1$, $\boldsymbol{\eta} = (k, \ldots, d')$ and h, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and a as before.

$\underline{n-s \leqslant d \leqslant n \text{ and } 0 \leqslant d' < k}$: In this case, the conditional distribution of $T_3^*/(n-r-s)$ given $D = d$, $D' = d'$ is the same as that of

$$\frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} + \sum_{i=1}^{d-n+s+1} \tilde{L}_{i:n-d}}{\sum_{i=d-n+s+2}^{d-r}(n-d+i-1)\tilde{L}_{i:d}},$$

This random variable has cdf $F_3(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda} = \varnothing, \boldsymbol{\mu} = \varnothing, a)$, see Proposition 5(d), with q the number of ties (pairs) in $\boldsymbol{\delta} = (n'-k+1, \ldots, n'-d', n-s, \ldots, d)$, $\boldsymbol{\theta}$ the vector of tied entries (each taken once), $\boldsymbol{\eta}$ the vector of untied entries, $p = k - d' + d - n + s + 1 - 2q$, and $a = n - r - s - 1$.

$\underline{n-s \leqslant d \leqslant n \text{ and } k \leqslant d' \leqslant n'}$: Here, the conditional distribution of $-T_3^*/(n-r-s)$, given $D = d$, $D' = d'$, is the same as that of

$$\frac{\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'} - \sum_{i=1}^{d-n+s+1} \tilde{L}_{i:n-d}}{\sum_{i=d-n+s+2}^{d-r}(n-d+i-1)\tilde{L}_{i:d}}.$$

This has cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda} = \varnothing, \boldsymbol{\mu} = \varnothing, a)$, see Proposition 4(d), with $p = d' - k + 1$, $\boldsymbol{\eta} = (k, \ldots, d')$ and $q = d - n + s + 1$, $\boldsymbol{\theta} = (n-s, \ldots, d)$ and $a = n - r - s - 1$.

Case $r \geqslant m$

By the symmetry of the standard Laplace distribution about zero, we have that $(X_{1:n}, \ldots, X_{n:n}) \overset{\mathrm{d}}{=} (-X_{n:n}, \ldots, -X_{1:n})$ and $Y_{k:n'} \overset{\mathrm{d}}{=} -Y_{n'-k+1:n'}$. It follows that

$$\begin{aligned} T_3(n,r,s,n',k) &= \frac{Y_{k:n'} - \hat{\mu}_{\mathrm{MLE}}}{\hat{\sigma}_{\mathrm{MLE}}} \\ &= \frac{Y_{k:n'} - X_{r+1:n}}{\left\{\sum_{i=r+1}^{n-s}(X_{i:n} - X_{r+1:n}) + s(X_{n-s:n} - X_{r+1:n})\right\}/(n-r-s)} + \log\left(\frac{n}{2(n-r)}\right) \\ &\overset{\mathrm{d}}{=} \frac{-Y_{n'-k+1:n'} + X_{n-r:n}}{\left\{\sum_{i=r+1}^{n-s}(X_{n-r:n} - X_{i:n}) + s(X_{n-r:n} - X_{s+1:n})\right\}/(n-r-s)} + \log\left(\frac{n}{2(n-r)}\right) \\ &\overset{\mathrm{d}}{=} -T_3(n,s,r,n',n'-k+1) \end{aligned}$$

by (2). Thus, the distribution of $-T_3(n,r,s,n',k)$ is obtained from the previous case after switching r with s and replacing k by $n'-k+1$.

Appendix

Lemma 2. *Let $U_1, \cdots, U_p \overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_p)$ be a vector of distinct positive numbers. Then the pdf of $\sum_{i=1}^{p} U_i/\eta_i$ is given by*

$$f(u;\boldsymbol{\eta}) = \sum_{j=1}^{p}\left(\prod_{\substack{i=1\\i\neq j}}^{p}\frac{\eta_i}{\eta_i-\eta_j}\right)\eta_j e^{-\eta_j u}, \quad u > 0.$$

Lemma 3. *Let $U_1, \ldots, U_p \overset{\text{iid}}{\sim} \mathcal{E}(1)$, $V_1, \ldots, V_q \overset{\text{iid}}{\sim} \mathcal{G}(2,1)$ be independent random variables and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_p)$, $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$ be vectors of distinct positive numbers. Then the pdf of $\sum_{j=1}^{p} U_j/\eta_j + \sum_{j=1}^{q} V_j/\theta_j$ is given by*

$$f(y;\boldsymbol{\eta},\boldsymbol{\theta}) = \sum_{j=1}^{p}\left(\prod_{\substack{i=1\\i\neq j}}^{p}\frac{\eta_i}{\eta_i-\eta_j}\right)\left(\prod_{i=1}^{q}\frac{\theta_i}{\theta_i-\eta_j}\right)^2\eta_j e^{-\eta_j y} +$$

$$\sum_{j=1}^{q}\left(\prod_{i=1}^{p}\frac{\eta_i}{\eta_i-\theta_j}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{q}\frac{\theta_i}{\theta_i-\theta_j}\right)^2\left\{y - \left(\sum_{i=1}^{p}\frac{1}{\eta_i-\theta_j} + \sum_{\substack{i=1\\i\neq j}}^{q}\frac{2}{\theta_i-\theta_j}\right)\right\}\theta_j^2 e^{-\theta_j y}, \quad y > 0.$$

Proof. See Jasiulewicz and Kordecki (2003). □

Proof of Proposition 3

We will prove here only part (a); the proofs of the other two parts are similar. Note first that $\sum_{j=1}^{q} v_j/\theta_j + w > 0$ for all $v_1, \ldots, v_q, w > 0$. Hence, for any $y > 0$ it holds

$$P(Y > y) = P\Big\{\sum_{j=1}^{p} U_j/\eta_j > \Big(\sum_{j=1}^{q} V_j/\theta_j + W\Big) y\Big\}$$
$$= \int_{v_1=0}^{\infty} \cdots \int_{v_q=0}^{\infty} \int_{w=0}^{\infty} \sum_{j=1}^{p} \Big(\prod_{\substack{i=1\\ i\neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j}\Big) e^{-\eta_j\{\sum_{i=1}^{q} v_i/\theta_i + w\}} e^{-\sum_{i=1}^{q} v_i} \frac{w^{a-1}e^{-w}}{\Gamma(a)} \mathrm{d}w\mathrm{d}v_q \cdots \mathrm{d}v_1$$

(by using Lemma 2)

$$= \sum_{j=1}^{p} \Big(\prod_{\substack{i=1\\ i\neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j}\Big) \Big(\prod_{i=1}^{q} \frac{\theta_i}{\theta_i + y\eta_j}\Big) \frac{1}{(1+y\eta_j)^a}.$$

Hence, the cdf equals one minus the above quantity as claimed.

Proof of Proposition 4

We will only prove (a); the remaining cases follow similarly. Before proceeding observe that the set B_2 of y's for which some denominator in the above formula becomes zero is finite. So, in what follows we will consider only $y \notin A_2 \cup B_2$. The result can be extended to this set as well by taking the limit due to the continuity of the cdf.

Observe first that

$$\begin{aligned} P(Y > y) = \; & P\Big\{\sum_{i=1}^{p} V_i/\eta_i - \sum_{i=1}^{q} U_i/\theta_i - \sum_{i=1}^{h} \lambda_i Z_i > \Big(\sum_{i=1}^{h} \mu_i Z_i + W\Big) y\Big\} \\ = \; & P\Big\{\sum_{i=1}^{p} V_i/\eta_i > \sum_{i=1}^{q} U_i/\theta_i + \sum_{i=1}^{h} (\lambda_i + y\mu_i) Z_i + yW\Big\}. \end{aligned}$$

Consider $y > 0$. Then, $\sum_{i=1}^{q} u_i/\theta_i + \sum_{i=1}^{h} (\lambda_i + y\mu_i) z_i + yw > 0$ for all positive v's, z's and w's. By using Lemma 2, this probability equals

$$\int_{\boldsymbol{u}} \int_{\boldsymbol{z}} \int_{w} \sum_{j=1}^{p} \Big(\prod_{\substack{i=1\\ i\neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j}\Big) e^{-\eta_j\{\sum_{i=1}^{q} u_i/\theta_i + \sum_{i=1}^{h}(\lambda_i + y\mu_i) z_i + yw\}} e^{-\sum_{i=1}^{q} u_i - \sum_{i=1}^{h} z_i} \frac{w^{a-1}e^{-w}}{\Gamma(a)} \mathrm{d}w\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{u}$$
$$= \sum_{j=1}^{p} \Big(\prod_{\substack{i=1\\ i\neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j}\Big) \Big(\prod_{i=1}^{q} \frac{1}{1 + \eta_j/\theta_i}\Big) \Big(\prod_{i=1}^{h} \frac{1}{1 + \eta_j(\lambda_i + y\mu_i)}\Big) \frac{1}{(1+y\eta_j)^a}.$$

It can be easily verified that $1 - P(Y > y)$ is as stated. Let now $y \in (-\rho_{\ell+1}, -\rho_\ell)$ for some $\ell \in \{0, \ldots, h-1\}$. Since the ratio λ_i/μ_i is strictly increasing in i, we have that

$\lambda_i + y\mu_i \leqslant 0$ for all $i \leqslant \ell$ and $\lambda_i + y\mu_i > 0$ for all $i > \ell$. Write then

$$P(Y \leqslant y) = P\Big\{ \sum_{i=1}^{p} V_i/\eta_i \leqslant \sum_{i=1}^{q} U_i/\theta_i + \sum_{i=1}^{h} \lambda_i Z_i + \Big(\sum_{i=1}^{h} \mu_i Z_i + W \Big) y \Big\}$$
$$= P\Big\{ \sum_{i=1}^{q} U_i/\theta_i + \sum_{i=\ell+1}^{h} (\lambda_i + y\mu_i) Z_i \geqslant \sum_{i=1}^{p} V_i/\eta_i - \sum_{i=1}^{\ell} (\lambda_i + y\mu_i) Z_i - yW \Big\}. \qquad (3)$$

Now, $\sum_{i=1}^{p} v_i/\eta_i - \sum_{i=1}^{\ell} (\lambda_i + y\mu_i) z_i - yw > 0$ for all positive v's, z's and w's. So, the last probability equals

$$\int_w \int_{\boldsymbol{z}} \int_{\boldsymbol{v}} \sum_{j=\ell+1}^{q+h} \Big(\prod_{\substack{i=\ell+1 \\ i \neq j}}^{q+h} \frac{\beta_i}{\beta_i - \beta_j} \Big) e^{-\beta_j \big(\sum\limits_{i=1}^{p} v_i/\eta_i - \sum\limits_{i=1}^{\ell} (\lambda_i + y\mu_i) z_i - yw \big)} e^{-\sum\limits_{i=1}^{p} v_i - \sum\limits_{i=1}^{\ell} z_i} \frac{w^{a-1} e^{-w}}{\Gamma(a)} \mathrm{d}\boldsymbol{v} \mathrm{d}\boldsymbol{z} \mathrm{d}w$$
$$= \sum_{j=\ell+1}^{q+h} \Big(\prod_{\substack{i=\ell+1 \\ i \neq j}}^{q+h} \frac{\beta_i}{\beta_i - \beta_j} \Big) \Big(\prod_{i=1}^{p} \frac{1}{1 + \beta_j/\eta_i} \Big) \Big(\prod_{i=1}^{\ell} \frac{1}{1 - (\lambda_i + y\mu_i)\beta_j} \Big) \frac{1}{(1 - y\beta_j)^a}.$$

Since for any $j = \ell + 1, \ldots, q + h$ it holds $\prod_{i=1}^{\ell} \frac{1}{1-(\lambda_i + y\mu_i)\beta_j} = \prod_{i=1}^{\ell} \frac{\beta_i}{\beta_i - \beta_j}$, the result follows. The proof for the case $y \in (-\infty, -\rho_h)$ proceeds similarly with the only difference that in (3) all Z's stay in the rhs.

Proof of Proposition 5

We only prove part (a); other cases can be proven similarly. Similarly to the proof of Proposition 4, we will consider only $y \notin A_3 \cup B_3$, where B_3 is the finite set of y's for which some denominator becomes zero. Again, the result can be extended to any $y \in (0, \infty)$ due to the continuity of the cdf.

For $\ell = 1, \ldots, h-1$ and $y \in (\rho_\ell, \rho_{\ell+1}) \backslash A_3 \cup B_3$, write

$$P(Y > y) = P\Big\{ \sum_{i=1}^{p} U_i/\eta_i + \sum_{i=1}^{q} V_i/\theta_i + \sum_{i=\ell+1}^{h} (\lambda_i - y\mu_i) Z_i > \sum_{i=1}^{\ell} (y\mu_i - \lambda_i) Z_i + yW \Big\}. \quad (4)$$

Since in this case $\lambda_i - y\mu_i < 0$ for $i = 1, \ldots, \ell$, it holds $\sum_{i=1}^{\ell} (y\mu_i - \lambda_i) z_i + yw > 0$ for all positive $z_1, \ldots, z_\ell, w$. Moreover, since and $\lambda_i - y\mu_i > 0$ for $i = \ell + 1, \ldots, h$, we may use Lemma 3 to get that the above probability equals

$$\int_w \int_{\boldsymbol{z}} \Big\{ \sum_{j=\ell+1}^{h+p} \Big(\prod_{\substack{i=1 \\ i \neq j}}^{h+p} \frac{\gamma_i}{\gamma_i - \gamma_j} \Big) \Big(\prod_{i=1}^{q} \frac{\theta_i}{\theta_i - \gamma_j} \Big)^2 e^{-\gamma_j \{ \sum_{i=1}^{\ell} (y\mu_i - \lambda_i) z_i + yw \}} +$$
$$\sum_{j=1}^{q} \Big(\prod_{i=\ell+1}^{h+p} \frac{\gamma_i}{\gamma_i - \theta_j} \Big) \Big(\prod_{\substack{i=1 \\ i \neq j}}^{q} \frac{\theta_i}{\theta_i - \theta_j} \Big)^2 e^{-\theta_j \{ \sum_{i=1}^{\ell} (y\mu_i - \lambda_i) z_i + yw \}} \times$$

$$\left(1+\theta_j\left[\sum_{i=1}^{\ell}(y\mu_i-\lambda_i)z_i+yw\right]-\sum_{i=\ell+1}^{p}\frac{\theta_j}{\gamma_i-\theta_j}-\sum_{\substack{i=1\\i\neq j}}^{q}\frac{2\theta_j}{\theta_i-\theta_j}\right)\Bigg\}\times$$

$$e^{-\sum_{i=1}^{\ell}z_i}\frac{w^{a-1}e^{-w}}{\Gamma(a)}\,\mathrm{d}\boldsymbol{z}\mathrm{d}w.$$

After integration and careful rearrangement of the terms, it can be verified that the last integral equals one minus the function stated in the proposition. The proofs for $y \in (0, \rho_1)$ [and $y \in (\rho_h, \infty)$] proceeds similarly with the only difference that in (4) all Z's stay in the lhs [rhs].

Proof of Proposition 6

The proof proceeds similarly to those of the previous propositions and therefore it is omitted.

Additional Reference

Jasiulewicz, H. and Kordecki, W. (2003). Convolutions of Erlang and of Pascal distributions with applications to reliability, *Demonstratio Mathematica*, **36**, 231–238.