Exact likelihood inference for Laplace distribution based on Type-II censored samples

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Abstract

We develop exact inference for the location and scale parameters of the Laplace (double exponential) distribution based on their maximum likelihood estimators from a Type-II censored sample. Based on some pivotal quantities, exact confidence intervals and tests of hypotheses are constructed. Upon conditioning first on the number of observations that are below the population median, exact distributions of the pivotal quantities are expressed as mixtures of linear combinations and of ratios of linear combinations of standard exponential random variables, which facilitates the computation of quantiles of these pivotal quantities. Tables of quantiles are presented for the complete sample case.

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1 Introduction

Let $X_1, \ldots, X_n$, $n \geqslant 2$, be a random sample from the Laplace (or double exponential) distribution $\mathcal{L}(\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$, with probability density function (pdf)

$$f(x; \mu, \sigma) = \frac{1}{2\sigma}\, e^{-\frac{1}{\sigma}\,|x-\mu|}, \quad x \in \mathbb{R}.$$

It is well-known (see, for example, Johnson et al., 1995, and Kotz et al., 2001) that the maximum likelihood estimators (MLEs) $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ are the sample median and the sample mean deviation from the sample median, respectively. Note that in the case of an even sample size, $\hat{\mu}$ is not unique since then any point between the two middle observations maximizes the likelihood function with respect to μ. However, it is customary to define the sample median as the average of the two middle observations and take that as the MLE of μ, and that is what we will do too

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hereafter. On the other hand, $\hat{\sigma}$ is always well-defined, since it turns out to be the difference of the sum of all sample points above $\hat{\mu}$ (whatever we take it to be) and the sum of all sample points below $\hat{\mu}$ divided by the sample size.

Balakrishnan and Cutler (1995) showed that the MLEs can be explicitly derived even in the presence of general Type-II censoring in the sample. Their result was further generalized by Childs and Balakrishnan (1997). To be more specific, let $X_{1:n} < \cdots < X_{n:n}$ denote the ordered sample and assume that r observations have been censored from the left and s observations have been censored from the right, i.e., the observed data consist of the order statistics

$$X_{r+1:n} < \cdots < X_{n-s:n},$$

which corresponds to a doubly Type-II censored sample. Such a censored sample may arise either naturally (for example, when some extreme values cannot be recorded due to experimental constraints or they are just missing) or intentionally (when the researcher decides to ignore some extreme observations based on robustness considerations). In order to be able to estimate both parameters, at least two observations are needed and so, we will assume that $n-r-s \geqslant 2$. Clearly, when $r = s = 0$, the complete data are observed. In what follows, $m = m(n) \geqslant 1$ is defined to be equal to $(n+1)/2$ when n is odd and $n/2$ when n is even. In other words, we are taking $n = 2m-1$ and $n = 2m$ for the odd and even sample cases, respectively. Then, from the above mentioned works, the following expressions of the MLEs are known:

- If $\max(r, s) < m$, then

$$\hat{\mu} = \begin{cases} X_{m:n}, & n = 2m-1 \\ \frac{1}{2}(X_{m:n} + X_{m+1:n}), & n = 2m, \end{cases}$$

  i.e., the sample median, and

$$\hat{\sigma} = \frac{1}{n-r-s}\left\{ \sum_{i=m+1}^{n-s} X_{i:n} + sX_{n-s:n} - rX_{r+1:n} - \sum_{i=r+1}^{[n/2]} X_{i:n} \right\},$$

  where $[x]$ denotes the integer part of x, and by convention $\sum_{i=k}^{\ell} \equiv 0$ when $k > \ell$;

- if $s \geqslant m$, then

$$\hat{\sigma} = \frac{1}{n-r-s}\left\{ \sum_{i=r+1}^{n-s} (X_{n-s:n} - X_{i:n}) + r(X_{n-s:n} - X_{r+1:n}) \right\}$$

  and

$$\hat{\mu} = X_{n-s:n} + \log\left(\frac{n/2}{n-s}\right)\hat{\sigma};$$

- if $r \geqslant m$, then

$$\hat{\sigma} = \frac{1}{n-r-s}\left\{\sum_{i=r+1}^{n-s}(X_{i:n} - X_{r+1:n}) + s(X_{n-s:n} - X_{r+1:n})\right\}$$

  and

$$\hat{\mu} = X_{r+1:n} + \log\left(\frac{n-r}{n/2}\right)\hat{\sigma}.$$

Since μ and σ are the location and scale parameters, respectively, it is evident that the random variables $T = (\hat{\mu} - \mu)/\hat{\sigma}$ and $S = \hat{\sigma}/\sigma$ have distributions which are free of both parameters and can therefore serve as pivotal quantities. Hence, inference about μ and σ can be carried out based on T and S, respectively. Indeed, Bain and Engelhardt (1973) considered approximate inference based on the above pivotal quantities. Kappenman (1975) subsequently developed conditional inference for μ and σ by conditioning on appropriate ancillary statistics. Grice, Bain and Engelhardt (1978) compared numerically the two approaches with respect to inference about μ and found that the conditional one gives slightly narrower confidence intervals. Childs and Balakrishnan (1996) extended Kappenman's (1975) conditional approach to the case of Type-II right censored data. A completely different procedure based on the distribution of the standard t-statistic in the Laplace case was considered by Sansing (1976).

In this paper, we develop exact inference for μ and σ based on T and S either when the sample is complete or when it is general Type-II censored. The importance of the results established here is two-fold. First, we provide the necessary tools for constructing exact confidence intervals and tests of hypotheses under the important Laplace model that often serves as an alternative to the normal distribution; see Kotz et al. (2001). Moreover, by tabulating the most commonly used quantiles of T and S (see Tables 4 and 5), we make these inferential processes as straightforward as in the case of normal samples. Secondly, exact inferential methods under censoring developed under Laplace model makes a substantial addition as there are very few models such as exponential, Pareto and uniform for which this development is possible.

The rest of this paper is organized as follows. In Section 2, we describe some known preliminary results on the Laplace distribution and also on the conditional distributions of order statistics which are essential for the ensuing developments. In Sections 3 and 4, we derive the exact distributions of S and T, respectively. We show that S is distributed as a mixture of linear combinations of independent standard exponential random variables while T is distributed as a mixture of ratios of (dependent) linear combinations of independent standard exponential random variables. Using these distributional results, in Section 5, we develop exact confidence intervals and tests of hypotheses about the parameters μ and σ. We also evaluate the performance of confidence intervals obtained from an asymptotic approach given by Bain and Engelhardt (1973) and from the parametric bootstrap approach. In Section 6, we use a real dataset as an example and illustrate all the inferential procedures developed here. In Section 7, we make some concluding remarks. Finally, an Appendix presents the derivations of all the above mentioned distributions of the pivotal quantities.

2 Preliminaries

Let $X \sim \mathcal{L}(\mu, \sigma)$. It is well-known that the parameter μ is the median of the distribution and thus, $\mathsf{P}(X \leqslant \mu) = \mathsf{P}(X \geqslant \mu) = 1/2$. Moreover, the conditional distribution of $X - \mu$, given $X > \mu$, is exponential with mean σ, denoted by $\mathcal{E}(\sigma)$. This is also the conditional distribution of $\mu - X$, given $X \leqslant \mu$.

Let $Z_1, \ldots, Z_n \overset{\text{iid}}{\sim} \mathcal{E}(\sigma)$ and denote by $Z_{1:n} < \cdots < Z_{n:n}$ the corresponding order statistics. For $i = 1, \ldots, n$, let us denote the i-th spacing $Z_{i:n} - Z_{i-1:n}$ by $\tilde{Z}_{i:n}$, where $Z_{0:n} \equiv 0$. Then, the normalized spacings

$$n\tilde{Z}_{1:n},\ (n-1)\tilde{Z}_{2:n},\ \cdots,\ (n-i+1)\tilde{Z}_{i:n}\,,\cdots,\ \tilde{Z}_{n:n}$$

form a random sample from $\mathcal{E}(\sigma)$ as well (see Arnold et al., 2008).

Iliopoulos and Balakrishnan (2009) established the following independence result concerning order statistics. If $X_1, \ldots, X_n$ is a random sample from any (either discrete or continuous) parent distribution and D denotes the number of X's being at most equal to a pre-fixed constant C, then, conditional on $D = d$, the blocks of order statistics $(X_{1:n}, \ldots, X_{d:n})$ and $(X_{d+1:n}, \ldots, X_{n:n})$ are independent. Moreover, conditional on $D = d$, $(X_{1:n}, \ldots, X_{d:n}) \overset{\text{d}}{=} (L_{1:d}, \ldots, L_{d:d})$ and $(X_{d+1:n}, \ldots, X_{n:n}) \overset{\text{d}}{=} (R_{1:n-d}, \ldots, R_{n-d:n-d})$, where $L_1, \ldots, L_d$ is a random sample from the parent distribution right truncated at C, and $R_1, \ldots, R_{n-d}$ is a random sample from the parent distribution left truncated at C.

Now, we shall explain how we can use the above results for the inferential problem at hand. Consider the complete sample $X_1, \ldots, X_n$ from $\mathcal{L}(\mu, \sigma)$ and set $D = \#\{X's \leqslant \mu\}$. Clearly, D follows a binomial $\mathcal{B}(n, 1/2)$ distribution. Conditional on $D = d$, $(\mu - X_{d:n}, \ldots, \mu - X_{1:n}) \overset{\text{d}}{=} (L_{1:d}, \ldots, L_{d:d})$ and $(X_{d+1:n} - \mu, \ldots, X_{n:n} - \mu) \overset{\text{d}}{=} (R_{1:n-d}, \ldots, R_{n-d:n-d})$, where $L_1, \ldots, L_d, R_1, \ldots, R_{n-d} \overset{\text{iid}}{\sim} \mathcal{E}(\sigma)$. Moreover, since any linear combination of $L_{i:d}$'s and $R_{j:n-d}$'s can be expressed as a linear combination of the spacings $\tilde{L}_{i:d}$'s and $\tilde{R}_{j:n-d}$'s which are independent $\mathcal{E}(\sigma)$ random variables, we will first condition on $D = d$ and express S and T through linear combinations of the above spacings. Then, we will derive the conditional distributions of S and T for all $d = 0, 1, \ldots, n$, and finally we will uncondition with respect to D in order to express the unconditional distributions of S and T as suitable mixtures. It should be mentioned that this mixture representation of the exact distributions of the MLEs, in the special case of complete samples of odd sizes, may be deduced from Proposition 2.6.6 of Kotz et al. (2001).

3 The distribution of $S = \hat{\sigma}/\sigma$

By invariance, the distribution of S does not depend on σ and so we may take $\sigma = 1$ without loss of any generality. In what follows, we set $D = \#\{X's \leqslant \mu\}$. Moreover, let $L_1, \ldots, L_n, R_1, \ldots, R_n$ be iid $\mathcal{E}(1)$ random variables and V an independent gamma $\mathcal{G}(a, 1)$ random variable with scale parameter 1 and shape parameter a which will be suitably determined later.

Case $\max(r, s) < m$

Consider first the case $\max(r, s) < m$. Depending on the value d of D we condition on, it is convenient to write $(n - r - s)S$ in different forms as follows:

- $d \leqslant r$: In this case, $(n - r - s)S$ can be expressed as

$$
\begin{aligned}
& r(\mu - X_{r+1:n}) + \sum_{i=r+1}^{[n/2]} (\mu - X_{i:n}) + \sum_{i=m+1}^{n-s} (X_{i:n} - \mu) + s(X_{n-s:n} - \mu) && (1)\\
& \overset{\mathrm{d}}{=} - r\, R_{r-d+1:n-d} - \sum_{i=r-d+1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} R_{i:n-d} + s\, R_{n-d-s:n-d} \\
& = \sum_{i=r-d+2}^{m-d} (d + i - 1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} (n - d - i + 1)\tilde{R}_{i:n-d} && (2)\\
& \overset{\mathrm{d}}{=} \sum_{i=r-d+2}^{m-d} \frac{d + i - 1}{n - d - i + 1} R_i + \sum_{i=m-d+1}^{n-d-s} R_i \overset{\mathrm{d}}{=} \sum_{i=r-d+2}^{m-d} \frac{d + i - 1}{n - d - i + 1} R_i + V,
\end{aligned}
$$

  where $V \sim \mathcal{G}(n - m - s, 1)$. It can be readily verified that in (1) the parameter μ is added as many times as it is subtracted and so it simply drops out. The same thing happens for all the cases that follow. Hence, the conditional pdf of $(n - r - s)S$, given $D = d \in \{0, \ldots, r\}$, is $f_S^{(m-r-1)}(x; \boldsymbol{\theta}, a)$ presented in Theorem 1, where $\boldsymbol{\theta} = \left(\frac{n-r-1}{r+1}, \ldots, \frac{n-m+1}{m-1}\right)$ and $a = n - m - s$.

- $r < d \leqslant m - 1$: In this case, we can express $(n - r - s)S$ as

$$
r(\mu - X_{r+1:n}) + \sum_{i=r+1}^{d} (\mu - X_{i:n}) - \sum_{i=d+1}^{[n/2]} (X_{i:n} - \mu) + \sum_{i=m+1}^{n-s} (X_{i:n} - \mu) + s(X_{n-s:n} - \mu)
$$

  (with the convention $\sum_{i=1}^{0} \equiv 0$ which occurs for $d = m - 1$ when $n = 2m - 1$)

$$
\overset{\mathrm{d}}{=} \left\{ \sum_{i=1}^{d-r} L_{i:d} + rL_{d-r:d} \right\} + \left\{ - \sum_{i=1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m+1-d}^{n-d-s} R_{i:n-d} + sR_{n-d-s:n-d} \right\}
$$

$$
\begin{aligned}
&=\Bigg\{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d}\Bigg\}+\Bigg\{\sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d}+\sum_{i=m-d+1}^{n-d-s}(n-d-i+1)\tilde{R}_{i:n-d}\Bigg\}\\
&\stackrel{\mathrm{d}}{=}\sum_{i=1}^{d-r}L_i+\sum_{i=1}^{m-d}\frac{d+i-1}{n-d-i+1}R_i+\sum_{i=m-d+1}^{n-d-s}R_i\stackrel{\mathrm{d}}{=}\sum_{i=1}^{m-d}\frac{d+i-1}{n-d-i+1}R_i+V,
\end{aligned}
$$

where $V\sim\mathcal{G}(n-m-s-r+d,1)$. Hence, $(n-r-s)S|(D=d)\sim f_S^{(m-d)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta}=\left(\frac{n-d}{d},\ldots,\frac{n-m+1}{m-1}\right)$ and $a=n-m-s-r+d$.

- $d=m\neq n-s$: In this case, we can express $(n-r-s)S$ as

$$
\begin{aligned}
&r(\mu-X_{r+1:n})+\sum_{i=r+1}^{[n/2]}(\mu-X_{i:n})+\sum_{i=m+1}^{n-s}(X_{i:n}-\mu)+s(X_{n-s:n}-\mu)\\
&\stackrel{\mathrm{d}}{=}\Bigg\{\sum_{i=m-[n/2]+1}^{m-r}L_{i:m}+r\,L_{m-r:m}\Bigg\}+\Bigg\{\sum_{i=1}^{n-m-s}R_{i:n-m}+sR_{n-m-s:n-m}\Bigg\}\\
&=\Bigg\{[n/2]L_{1:m}+\sum_{i=2}^{m-r}(m-i+1)\tilde{L}_{i:m}\Bigg\}+\Bigg\{\sum_{i=1}^{n-m-s}(n-m-i+1)\tilde{R}_{i:n-m}\Bigg\}\\
&\stackrel{\mathrm{d}}{=}\frac{[n/2]}{m}L_1+\sum_{i=2}^{m-r}L_i+\sum_{i=1}^{n-m-s}R_i\stackrel{\mathrm{d}}{=}\frac{[n/2]}{m}L_1+V,
\end{aligned}
$$

where $V\sim\mathcal{G}(n-s-r-1,1)$. Note that when $n=2m$ the above conditional distribution becomes $\mathcal{G}(n-s-r,1)$. Thus, $(n-r-s)S|(D=m)\sim f_S^{(1)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta}=\left(\frac{m}{m-1}\right)$ and $a=n-s-r-1$ when $n=2m-1$, or $\mathcal{G}(n-s-r,1)$ when $n=2m$.

- $m+1\leqslant d<n-s$: Due to symmetry, $(n-r-s)S$ has the same conditional distribution either when we condition on $D=d$ or on $D=n-d$. So, by interchanging d and $n-d$ in the conditional distribution of the case $r<d\leqslant m-1$, we have $(n-r-s)S|(D=d)\sim f_S^{(m-n+d)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta}=\left(\frac{d}{n-d},\cdots,\frac{n-m+1}{m-1}\right)$ and $a=2n-m-r-s-d$.

- $n-s\leqslant d\leqslant n$: Yet again, due to symmetry, the conditional distribution of $(n-r-s)S$, given $D=d$, is $f_S^{(m-s-1)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta}=\left(\frac{n-s-1}{s+1},\cdots,\frac{n-m+1}{m-1}\right)$ and $a=n-m-r$.

Case $s\geqslant m$

- $d\leqslant r$: In this case, $(n-r-s)S$ may be expressed as

$$
\begin{aligned}
&-r(X_{r+1:n}-\mu)-\sum_{i=r+1}^{n-s}(X_{i:n}-\mu)+(n-s)(X_{n-s:n}-\mu)\\
&\stackrel{\mathrm{d}}{=}-r\,R_{r-d+1:n-d}-\sum_{i=r-d+1}^{n-d-s}R_{i:n-d}+(n-s)R_{n-d-s:n-d}
\end{aligned}
$$

$$= \sum_{i=r-d+2}^{n-d-s} (d+i-1)\tilde{R}_{i:n-d} \overset{\mathrm{d}}{=} \sum_{i=r-d+2}^{n-d-s} \frac{d+i-1}{n-d-i+1} R_i.$$

Hence, $(n-r-s)S|(D=d) \sim f_S^{(n-r-s-1)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta} = \left(\frac{n-r-1}{r+1}, \cdots, \frac{s+1}{n-s-1}\right)$ and $a=0$.

- $r+1 \leqslant d < n-s$: In this case, we can express $(n-r-s)S$ as

$$\begin{aligned}
&r(\mu - X_{r+1:n}) + \sum_{i=r+1}^{d} (\mu - X_{i:n}) - \sum_{i=d+1}^{n-s} (X_{i:n}-\mu) + (n-s)(X_{n-s:n}-\mu) \\
&\overset{\mathrm{d}}{=} \left\{ \sum_{i=1}^{d-r} L_{i:d} + rL_{d-r:d} \right\} + \left\{ -\sum_{i=1}^{n-d-s} R_{i:n-d} + (n-s)R_{n-d-s:n-d} \right\} \\
&= \left\{ \sum_{i=1}^{d-r} (d-i+1)\tilde{L}_{i:d} \right\} + \left\{ \sum_{i=1}^{n-d-s} (d+i-1)\tilde{R}_{i:n-d} \right\} \\
&\overset{\mathrm{d}}{=} \sum_{i=1}^{d-r} L_i + \sum_{i=1}^{n-d-s} \frac{d+i-1}{n-d-i+1} R_i \overset{\mathrm{d}}{=} \sum_{i=1}^{n-d-s} \frac{d+i-1}{n-d-i+1} R_i + V,
\end{aligned}$$

where $V \sim \mathcal{G}(d-r,1)$. Hence, $(n-r-s)S|(D=d) \sim f_S^{(n-d-s)}(x;\boldsymbol{\theta},a)$, where $\boldsymbol{\theta} = \left(\frac{n-d}{d}, \cdots, \frac{s+1}{n-s-1}\right)$ and $a = d-r$.

- $d \geqslant n-s$: Finally, in this case, $(n-r-s)S$ can be expressed as

$$\begin{aligned}
&r(\mu - X_{r+1:n}) + \sum_{i=r+1}^{n-s} (\mu - X_{i:n}) - (n-s)(\mu - X_{n-s:n}) \\
&\overset{\mathrm{d}}{=} rL_{d-r:d} + \sum_{i=d-n+s+1}^{d-r} L_{i:d} - (n-s)L_{d-n+s+1:d} \\
&= \sum_{i=d-n+s+2}^{d-r} (d-i+1)\tilde{L}_{i:d} \overset{\mathrm{d}}{=} \sum_{i=1}^{n-r-s-1} L_i \overset{\mathrm{d}}{=} V,
\end{aligned}$$

where $V \sim \mathcal{G}(n-r-s-1,1)$. Hence, $(n-r-s)S|(D=d) \sim \mathcal{G}(n-r-s-1,1)$.

Case $\boldsymbol{r \geqslant m}$

Using again a symmetry argument, we conclude that the conditional distributions of $(n-r-s)S$, given $D=d$, are as in the previous section when interchanging r and s, and replacing d by $n-d$.

Let $f^*_{S|D}(x|d)$ denote in general the conditional pdf of $(n-r-s)S$, given $D=d$. Then, by standard arguments, the conditional pdf of S, given $D=d$, is $f_{S|D}(x|d) = (n-r-s)f^*_{S|D}((n-r-s)x|d)$. Using now all the conditional pdfs of S, given $D=d$, presented above, we can express the exact pdf of S as

$$f_S(x) = \sum_{d=0}^{n} \mathsf{P}(D=d) f_{S|D}(x|d) = \frac{1}{2^n} \sum_{d=0}^{n} \binom{n}{d} f_{S|D}(x|d), \quad x>0.$$

Note also that the distribution of S remains the same when the values of r and s are interchanged.

4 The distribution of $T = (\hat{\mu} - \mu)/\hat{\sigma}$

Let $\hat{\mu}$ be the sample median, that is,

$$\hat{\mu} = \begin{cases} X_{m:n}, & n = 2m-1 \\ \frac{1}{2}(X_{m:n} + X_{m+1:n}), & n = 2m \end{cases} = \begin{cases} \sum_{i=1}^{m} \tilde{X}_{i:n}, & n = 2m-1 \\ \sum_{i=1}^{m} \tilde{X}_{i:n} + \frac{1}{2}\tilde{X}_{m+1:n}, & n = 2m. \end{cases}$$

Once again, by invariance, we may take $\sigma = 1$ without loss of any generality. In what follows, U's, Z's, and W denote independent random variables, where U's and Z's $\overset{\text{iid}}{\sim} \mathcal{E}(1)$ while W is a gamma random variable with scale parameter 1 and shape parameter that will be suitably determined. Moreover, the expressions for the conditional pdfs of T, given $D = d$, are presented in Theorems 2–6.

Case $\max(r, s) < m$

- $d \leqslant r$: By conditioning on $D = d$, we may write

$$\begin{aligned} \hat{\mu} - \mu &\overset{\text{d}}{=} \begin{cases} R_{m-d:n-d}, & n = 2m-1 \\ \frac{1}{2}(R_{m-d:n-d} + R_{m+1-d:n-d}), & n = 2m \end{cases} \\ &= \begin{cases} \sum_{i=1}^{m-d} \tilde{R}_{i:n-d}, & n = 2m-1 \\ \sum_{i=1}^{m-d} \tilde{R}_{i:n-d} + \frac{1}{2}\tilde{R}_{m+1-d:n-d}, & n = 2m. \end{cases} \end{aligned} \tag{3}$$

  Upon using (2), we can see that when $n = 2m-1$, conditional on $D = d$, $T/(n-r-s) \equiv (\hat{\mu} - \mu)/\{(n-r-s)\hat{\sigma}\}$ has the same distribution as

$$\frac{\sum\limits_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum\limits_{i=r-d+2}^{m-d} (d+i-1)\tilde{R}_{i:n-d} + \sum\limits_{i=m-d+1}^{n-d-s} (n-d-i+1)\tilde{R}_{i:n-d}}$$

$$\overset{\text{d}}{=} \frac{\sum\limits_{i=1}^{r-d+1} \frac{1}{n-d-i+1} R_i + \sum\limits_{i=r-d+2}^{m-d} \frac{1}{n-d-i+1} R_i}{\sum\limits_{i=r-d+2}^{m-d} \frac{d+i-1}{n-d-i+1} R_i + \sum\limits_{i=m-d+1}^{n-d-s} R_i} \overset{\text{d}}{=} \frac{\sum\limits_{i=1}^{r-d+1} \frac{1}{n-d-i+1} U_i + \sum\limits_{i=1}^{m-r-1} \frac{1}{m+i-1} Z_i}{\sum\limits_{i=1}^{m-r-1} \frac{m-i}{m+i-1} Z_i + W},$$

  where $W \sim \mathcal{G}(n-m-s, 1) \equiv \mathcal{G}(m-s-1, 1)$. Hence, $T/(n-r-s)|(D = d) \sim f_T^{(r-d+1, m-r-1)}(x; \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$, where $\boldsymbol{\theta} = (n-r, \cdots, n-d)$, $\boldsymbol{\lambda} = \left(\frac{1}{m}, \cdots, \frac{1}{n-r-1}\right)$, $\boldsymbol{\mu} = \left(\frac{m-1}{m}, \cdots, \frac{r+1}{n-r-1}\right)$ and $a = m-s-1$. On the other hand, when $n = 2m$, we have, conditional on $D = d$, $T/(n-r-s)$ to have the same distribution as

$$\frac{\sum\limits_{i=1}^{m-d} \tilde{R}_{i:n-d} + \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum\limits_{i=r-d+2}^{m-d} (d+i-1)\tilde{R}_{i:n-d} + \sum\limits_{i=m-d+1}^{n-d-s} (n-d-i+1)\tilde{R}_{i:n-d}}$$

$$\overset{\mathrm{d}}{=} \frac{\sum_{i=1}^{r-d+1} \frac{1}{n-d-i+1} R_i + \sum_{i=r-d+2}^{m-d} \frac{1}{n-d-i+1} R_i + \frac{1}{2m} R_{m-d+1}}{\sum_{i=r-d+2}^{m-d} \frac{d+i-1}{n-d-i+1} R_i + R_{m-d+1} + \sum_{i=m-d+2}^{n-d-s} R_i}$$

$$\overset{\mathrm{d}}{=} \frac{\sum_{i=1}^{r-d+1} \frac{1}{n-d-i+1} U_i + \frac{1}{2m} Z_1 + \sum_{i=2}^{m-r} \frac{1}{m+i-1} Z_i}{\sum_{i=1}^{m-r} \frac{m-i+1}{m+i-1} Z_i + W},$$

where $W \sim \mathcal{G}(n-m-s-1,1) \equiv \mathcal{G}(m-s-1,1)$. Hence, $T/(n-r-s)|(D=d) \sim f_T^{(r-d+1,m-r)}(x;\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$, with $\boldsymbol{\theta}$ and a as before and $\boldsymbol{\lambda} = \left(\frac{1}{2m}, \frac{1}{m+1}, \cdots, \frac{1}{n-r-1}\right)$ and $\boldsymbol{\mu} = \left(1, \frac{m-1}{m+1}, \cdots, \frac{r+1}{n-r-1}\right)$.

- $r < d \leqslant m-1$: In this case, the conditional distribution of $\hat{\mu}-\mu$, given $D=d$, is once again as in (3). Thus, when $n = 2m-1$, we have, conditional on $D=d$, $T/(n-r-s)$ to have the same distribution as

$$\frac{\sum_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-r-s}(n-d-i+1)\tilde{R}_{i:n-d}} \overset{\mathrm{d}}{=} \frac{\sum_{i=1}^{m-d} \frac{1}{m+i-1} Z_i}{\sum_{i=1}^{m-d} \frac{m-i}{m+i-1} Z_i + W},$$

where $W \sim \mathcal{G}(n-r-s-m+d,1) \equiv \mathcal{G}(m-r-s+d-1,1)$. Hence, $T/(n-r-s)|(D=d) \sim f_T^{(0,m-d)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$, where $\boldsymbol{\lambda} = \left(\frac{1}{m}, \cdots \frac{1}{n-d}\right)$, $\boldsymbol{\mu} = \left(\frac{m-1}{m}, \cdots, \frac{d}{n-d}\right)$ and $a = m-r-s+d-1$. When $n = 2m$, $T/(n-r-s)$ has the same distribution as

$$\frac{\sum_{i=1}^{m-d} \tilde{R}_{i:n-d} + \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-r-s}(n-d-i+1)\tilde{R}_{i:n-d}} \overset{\mathrm{d}}{=} \frac{\frac{1}{2m} Z_1 + \sum_{i=2}^{m-d+1} \frac{1}{m+i-1} Z_i}{\sum_{i=1}^{m-d+1} \frac{m-i+1}{m+i-1} Z_i + W},$$

where $W \sim \mathcal{G}(n-r-s-m+d-1,1) \equiv \mathcal{G}(m-r-s+d-1,1)$. Hence, $T/(n-r-s)|(D=d) \sim f_T^{(0,m-d+1)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$, where $\boldsymbol{\lambda} = \left(\frac{1}{2m}, \frac{1}{m+1}, \cdots, \frac{1}{n-d}\right)$, $\boldsymbol{\mu} = \left(1, \frac{m-1}{m+1}, \cdots, \frac{d}{n-d}\right)$ and $a = m-r-s+d-1$.

- $d = m \neq n-s$: In this case, the conditional distribution of $\hat{\mu}-\mu$, given $D=m$, is

$$\hat{\mu}-\mu \overset{\mathrm{d}}{=} \begin{cases} -L_{1:m}, & n = 2m-1 \\ \frac{1}{2}(R_{1:n-m} - L_{1:m}), & n = 2m. \end{cases}$$

Thus, when $n = 2m-1$, we have, conditional on $D = m$, $T/(n-r-s)$ has the same distribution as $-\frac{1}{m} Z_1 / \left\{\frac{m-1}{m} Z_1 + W\right\}$, where $W \sim \mathcal{G}(n-r-s-1,1) \equiv \mathcal{G}(2m-r-s-2,1)$, that is, $-T/(n-r-s)|(D=d) \sim f_T^{(0,1)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $\boldsymbol{\lambda} = \left(\frac{1}{m}\right)$, $\boldsymbol{\mu} = \left(\frac{m-1}{m}\right)$ and $a = 2m-r-s-2$. On the other hand, when $n = 2m$, $T/(n-r-s) \overset{\mathrm{d}}{=} \frac{1}{2m}(Z_1 - Z_2)/(Z_1 + Z_2 + W)$, where

$W \sim \mathcal{G}(n-r-s-2,1) \equiv \mathcal{G}(2m-r-s-2,1)$. Thus, in this case, $T/(n-r-s)|(D=d) \sim g_T(x;1/2m, 2m-r-s-2)$ as given in Theorem 6.

- $m+1 \leqslant d < n-s$: Using a symmetry argument, we get when $n = 2m-1$, $-T/(n-r-s)|(D=d) \sim f_T^{(0,d-m+1)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$, where $\boldsymbol{\lambda} = \left(\frac{1}{m},\cdots,\frac{1}{d}\right)$, $\boldsymbol{\mu} = \left(\frac{m-1}{m},\cdots,\frac{n-d}{d}\right)$ and $a = n-r-s-d+m-1$, while when $n=2m$, $-T/(n-r-s)|(D=d) \sim f_T^{(0,d-m+1)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$, where $\boldsymbol{\lambda} = \left(\frac{1}{2m},\frac{1}{m+1},\cdots,\frac{1}{d}\right)$, $\boldsymbol{\mu} = \left(1,\frac{m-1}{m+1},\cdots,\frac{n-d}{d}\right)$ and $a = n-r-s-d+m-1$.

- $d \geqslant n-s$: Yet again, by a symmetry argument, we get when $n=2m-1$, $-T/(n-r-s)|(D=d) \sim f_T^{(d-n+s+1,m-s-1)}(x;\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $\boldsymbol{\theta} = (n-s,\cdots,d)$, $\boldsymbol{\lambda} = \left(\frac{1}{m},\cdots,\frac{1}{n-s-1}\right)$, $\boldsymbol{\mu} = \left(\frac{m-1}{m},\cdots,\frac{s+1}{n-s-1}\right)$ and $a = m-r-1$, while when $n=2m$, $-T/(n-r-s)|(D=d) \sim f_T^{(d-n+s+1,m-s)}(x;\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $\boldsymbol{\theta}$ and a as before, $\boldsymbol{\lambda} = \left(\frac{1}{2m},\frac{1}{m+1},\cdots,\frac{1}{n-s-1}\right)$ and $\boldsymbol{\mu} = \left(1,\frac{m-1}{m+1},\cdots,\frac{s+1}{n-s-1}\right)$.

Now, let $f^*_{T|D}(x|d)$ denote in general the conditional pdf of $T/(n-r-s)$, given $D=d$. Then, by standard arguments, $T|(D=d) \sim f_{T|D}(x|d) = (n-r-s)^{-1} f^*_{T|D}(x/(n-r-s)|d)$ and the exact pdf of T is given by

$$f_T(x) = \frac{1}{2^n}\sum_{d=0}^{n}\binom{n}{d} f_{T|D}(x|d), \quad x \in \mathbb{R}.$$

Case $s \geqslant m$

Note here that

$$\frac{\hat{\mu}-\mu}{\hat{\sigma}} = \frac{X_{n-s:n}-\mu}{\hat{\sigma}} + \log\left(\frac{n/2}{n-s}\right)$$

and so, we actually need the conditional distributions of $T^* = (X_{n-s:n}-\mu)/\hat{\sigma}$.

- $d \leqslant r$: Conditional on $D=d$, $T^*/(n-r-s)$ has the same distribution as

$$\frac{\sum\limits_{i=1}^{n-d-s}\tilde{R}_{i:n-d}}{\sum\limits_{i=r-d+2}^{n-d-s}(d+i-1)\tilde{R}_{i:n-d}} \stackrel{\mathrm{d}}{=} \frac{\sum\limits_{i=1}^{r-d+1}\frac{1}{n-d-i+1}U_i + \sum\limits_{i=1}^{n-r-s-1}\frac{1}{s+i}Z_i}{\sum\limits_{i=1}^{n-r-s-1}\frac{n-s-i}{s+i}Z_i},$$

  and hence, $T^*/(n-r-s)|(D=d) \sim f_T^{(r-d+1,n-r-s-1)}(x;\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},0)$ as given in Theorem 4, where $\boldsymbol{\theta} = (n-r,\cdots,n-d)$, $\boldsymbol{\lambda} = \left(\frac{1}{n-r-1},\cdots,\frac{1}{s+1}\right)$ and $\boldsymbol{\mu} = \left(\frac{r+1}{n-r-1},\cdots,\frac{n-s-1}{s+1}\right)$.

- $r+1 \leqslant d < n-s$: Conditional on $D=d$, $T^*/(n-r-s)$ has the same distribution as

$$\frac{\sum\limits_{i=1}^{n-d-s}\tilde{R}_{i:n-d}}{\sum\limits_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum\limits_{i=1}^{n-d-s}(d+i-1)\tilde{R}_{i:n-d}} \stackrel{\mathrm{d}}{=} \frac{\sum\limits_{i=1}^{n-d-s}\frac{1}{s+i}Z_i}{\sum\limits_{i=1}^{n-d-s}\frac{n-s-i}{s+i}Z_i + W},$$

where $W \sim \mathcal{G}(d-r,1)$. Thus, $T^*/(n-r-s)|(D=d) \sim f_T^{(0,n-r-s-1)}(x;\boldsymbol{\lambda},\boldsymbol{\mu},a)$, where $\boldsymbol{\lambda} = \left(\frac{1}{s+1},\cdots,\frac{1}{n-d}\right)$, $\boldsymbol{\mu} = \left(\frac{n-s-1}{s+1},\cdots,\frac{d}{n-d}\right)$ and $a = d-r$.

- $d \geqslant n-s$: Conditional on $D=d$, $T^*/(n-r-s)$ has the same distribution as

$$-\frac{\sum_{i=1}^{d-n+s+1}\tilde{L}_{i:d}}{\sum_{i=d-n+s+2}^{d-m+1}(n-d+i-1)\tilde{L}_{i:d}} \stackrel{\mathrm{d}}{=} -\frac{\sum_{i=1}^{d-n+s+1}\frac{1}{d-i+1}U_i}{W}$$

where $W \sim \mathcal{G}(n-r-s-1,1)$. Thus, $-T^*/(n-r-s)|(D=d) \sim f_T^{(d-n+s+1,0)}(x;\boldsymbol{\theta},a)$ as given in Theorem 5, where $\boldsymbol{\theta} = (n-s,\cdots,d)$ and $a = n-r-s-1$. It should be noted that this is the only case in which the conditional distribution can be written as the ratio of two independent random variables.

Now, let $f^*_{T|D}(x|d)$ denote the conditional distribution of $T^*/(n-r-s)$, given $D=d$, and let $K = \log\left(\frac{n/2}{n-s}\right)$. Then, $T|(D=d) \sim f_{T|D}(x|d) = (n-r-s)^{-1}f^*_{T|D}((x-K)/(n-r-s)|d)$ and the exact pdf of T is given once again by

$$f_T(x) = \frac{1}{2^n}\sum_{d=0}^{n}\binom{n}{d}f_{T|D}(x|d), \quad x \in \mathbb{R}.$$

Case $\boldsymbol{r \geqslant m}$

By symmetry, the conditional distributions can be deduced from the previous case.

5 Exact inference and comparison with asymptotic approach

Since S and T are pivotal quantities for σ and μ, respectively, they can be used for developing exact inferential procedures for the two parameters. Their forms are analogous to that of the familiar normal theory involving chi-square and t distributed pivots and so everything works in exactly the same manner. For instance, let $S_{n,r,s;\alpha}$ and $T_{n,r,s;\alpha}$ denote the upper α-quantiles of S and T when the sample size equals n and r and s observations have been censored from the left and right, respectively. Then, a $100(1-\alpha)\%$ exact confidence interval for σ is $[\hat{\sigma}/S_{n,r,s;\alpha/2}, \hat{\sigma}/S_{n,r,s;1-\alpha/2}]$, while the null hypothesis $\sigma = \sigma_0$ will be rejected in favor of the alternatives $\sigma > \sigma_0$, $\sigma < \sigma_0$ or $\sigma \neq \sigma_0$ if $\hat{\sigma}/\sigma_0$ is larger than $S_{n,r,s;\alpha}$, smaller than $S_{n,r,s;1-\alpha}$, or outside interval $[S_{n,r,s,1-\alpha/2}, S_{n,r,s,\alpha/2}]$, respectively. In a similar vein, a $100(1-\alpha)\%$ exact confidence interval for μ is given by $[\hat{\mu} - T_{n,r,s;\alpha/2}\hat{\sigma}, \hat{\mu} - T_{n,r,s;1-\alpha/2}\hat{\sigma}]$, and testing the hypothesis $\mu = \mu_0$ against the usual one- or two-sided alternatives is then carried out in the usual manner. Note here that unless $r = s$ wherein the distribution of T is symmetric about the origin, $T_{n,r,s;1-\alpha/2} \neq -T_{n,r,s;\alpha/2}$.

In order to find any required quantile, one needs to solve a nonlinear equation. Although the exact pdfs of S and T look quite cumbersome, the task can be accomplished by using an appropriate strategy. First, from Theorem 1, one can see that in order to calculate quantiles of S, the lower incomplete gamma function is needed. This function is readily available in almost any statistics or mathematics package and can be accurately evaluated. Next, even though the conditional cdfs' of T in Theorem 2 appear to be more difficult to compute, observe that since a is integer-valued, both numerator and denominator of each term of the sum consists of factorized polynomials. Hence, the method of partial fractions can give the required result. With respect to the accuracy of calculations, the fact that $\boldsymbol{\theta}$'s, $\boldsymbol{\lambda}$'s and $\boldsymbol{\mu}$'s are vectors of either integers or rationals allows us to work only with rational numbers and thus achieve any desired precision. In fact, this is exactly what we did for determining the quantiles of T using `Mathematica`. In Tables 4 and 5, we have tabulated quantiles of the exact distributions of T and S, respectively, for n up to 40 in the case of complete samples. More tables can be found at `http:\\www.unipi.gr\faculty\geh\Quantiles.zip`

In the case of complete samples, Bain and Engelhardt (1973) relied on approximations of the distributions of S and T in order to construct confidence intervals and tests of hypotheses for the Laplace parameters. In fact, they started by providing their exact distributions when $n = 3$ and 5, but they then stated that "'the derivation (...) becomes quite tedious as n increases". As we have derived here the exact distributions of S and T for all sample sizes, it would be worthwhile to evaluate the actual coverage probability of the approximate confidence intervals proposed by these authors.

According to Bain and Engelhardt (1973), the most convenient approach for constructing a confidence interval for σ is based on the approximation of the distribution of $2nS$ by a chi-square distribution with $2n\mathsf{E}(S)$ degrees of freedom. By using the exact distribution of S, we observed that the above chi-square distribution provides a very good approximation indeed. More specifically, for $n \geqslant 10$, the actual confidence coefficients of the approximated intervals equal nominal values of 90%, 95% and 99% when rounded to the third decimal place.

In order to construct confidence intervals for μ, Bain and Engelhardt (1973) considered several approaches. One of these is based on the fact that $T^* = n^{1/2}(\hat{\mu} - \mu)/\sigma$ and $S^* = n^{1/2}(\hat{\sigma}/\sigma - 1)$ have asymptotic standard normal distributions (see also Chernoff et al., 1967). Since $\hat{\sigma}$ is a consistent estimator of σ, Slutsky's Lemma ensures that $n^{1/2}T$ has an asymptotic standard normal distribution as well. However, the confidence intervals obtained by using the latter normal approximation are too narrow for finite samples and hence they have considerably less coverage probability than the nominal level. Therefore, they suggested to exploit the fact that, for finite samples, $\hat{\mu}$, $\hat{\sigma}$ are uncorrelated which implies that S^*, T^* are asymptotically independent. Then, an application of the delta method shows that $n^{1/2}T/(1+T^2)^{1/2}$ also has an asymptotic standard normal distribution. This in turns implies that an asymptotic $100(1-\alpha)\%$ confidence interval for μ is given by $\hat{\mu} \pm \hat{\sigma} z_{\alpha/2}/(n - z^2_{\alpha/2})^{1/2}$, where z_α denotes the upper α-quantile of the standard normal distribution. However, the exact coverage probability of this interval is quite below the nominal confidence level as can be seen in Table 1.

Another simple strategy for constructing confidence intervals for the Laplace parameters is the parametric boostrap approach, i.e., Monte Carlo sampling from $\mathcal{L}(\hat{\mu}, \hat{\sigma})$. Table 2 and 3 contain results of a simulation study on the coverage probabilities of boostrap confidence intervals for the two parameters. The results are based on 10000 simulations with the number of boostrap samples taken to be 1000. Note here that for the confidence interval for σ the bias-corrected and accelerated percentile method (see Efron and Tibshirani, 1982, pp. 184-188) has been used since the distribution of $\hat{\sigma}$ is asymmetric. As we can see the estimated confidence levels are quite close to the nominal ones for small sample sizes and achieve them for moderate sample sizes. However, this approach does not differ much from estimating the quantiles of S and T by Monte Carlo which is an easy task since they are pivotal quantities.

In conclusion, the distribution of $2nS$ can be approximated very well by a particular chi-square distribution and so, the latter can be used in order to avoid solving the rather cumbersome equations that yield the corresponding exact quantiles. On the other hand, the normal approximation of the distribution of T is poor (at least for moderate sample sizes) which means that its exact distribution is necessary for exact inference about the location parameter. Furthermore, the parametric bootstrap approach seem to work well provided that the number of bootstrap samples is large. In any case, for the convenience of users, we have provided in Tables 4 and 5 the most important quantiles of both exact distributions that would facilitate exact inference when dealing with Laplace distribution.

6 An illustrative example

Bain and Engelhardt (1973) considered 33 years of flood data from two stations on Fox River, Wisconsin. They modeled the data using a Laplace distribution and provided 95% approximate confidence intervals (c.i.'s) for the location and scale parameters based on the pivotal quantities T and S, respectively. Kappenman (1975) analyzed further these data for illustrating his conditional approach. The data are presented in Table 6. Here we provide 95% exact c.i.'s using the distributions of T and S presented in the preceding sections.

From the data, we find $\hat{\mu} = 10.13$ and $\hat{\sigma} = 3.36091$. From Table 4, we see that the 0.025-quantile of T is 0.4128, and so the exact 95% c.i. for the location parameter μ is

$$[10.13 - 0.4128 \times 3.36091, 10.13 + 0.4128 \times 3.36091] = [8.74, 11.52].$$

For comparative purposes, we note that Bain and Engelhardt gave the approximate 95% c.i. for μ to be $[8.91, 11.35]$, while Kappenman's conditional approach yielded $[8.99, 12.41]$. For a c.i. for the scale parameter σ, we find from Table 5 the 0.975 and 0.025 quantiles of S to be 1.3492 and 0.6745, respectively. So, the 95% equi-tailed c.i. for σ is

$$[3.36091/1.3492, 3.36091/0.6745] = [2.49, 4.98].$$

This essentially agrees with Bain and Engelhardt's approximate c.i. and Kappenman's conditional c.i. of $[2.49, 4.97]$.

Childs and Balakrishnan (1996) discussed conditional inference of Laplace parameters under Type-II right censoring. As an example, they considered the Fox River flood data and assumed that the 10 largest observations had been censored. They reported the 95% conditional c.i.'s for μ and σ to be $[7.69, 11.40]$ and $[2.73, 6.30]$, respectively. Using the distributions derived in the previous sections, we found the 0.975 and 0.025 quantiles of T to be -0.4193 and 0.4191, respectively. (Note that in this case the distribution of T is not symmetric due to the unbalanced censoring and that's why the two quantiles differ in absolute value. Their absolute values are, however, too close because of the large sample size.) In this case, we have $\hat{\mu} = 10.13$, $\hat{\sigma} = 3.88217$ and so, the 95% exact c.i. for μ is

$$[10.13 - 0.4191 \times 3.88217, 10.13 + 0.4193 \times 3.88217] = [8.50, 11.76].$$

Similarly, we found the 0.975 and 0.025 quantiles of S in this case to be 1.4190 and 0.6147, respectively. So, the exact 95% c.i. for σ is

$$[3.88217/1.4190, 3.88217/0.6147] = [2.74, 6.32]$$

which, as in the case of complete sample, essentially coincides with that obtained by the conditional approach.

7 Concluding remarks

Here, we have developed exact distributional results for the distributions of the pivotal quantities based on the MLEs of the location and scale parameters of Laplace distribution based on general Type-II censored samples. It would be possible to develop similar results when the location and scale parameters μ and σ are estimated by other L-estimators such as the best linear unbiased estimators and best linear invariant estimators. Also, the results developed here for the case of Type-II censoring could be adopted to the situation when the available sample is progressively Type-II censored (see Balakrishnan, 2007, for details) or Type-I censored. Work on these problems is currently under progress and we hope to report these findings in a future paper.

Appendix

Lemma 1. *Let $U_1, \ldots, U_k \overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ be a vector of distinct positive real numbers. Then, the pdf of $U = \sum_{i=1}^{k} U_i/\theta_i$ is given by*

$$f_U(u; \boldsymbol{\theta}) = \sum_{j=1}^{k} \Bigg(\prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{\theta_i}{\theta_i - \theta_j} \Bigg) \theta_j e^{-u\theta_j}, \quad u > 0.$$

Theorem 1. *Let $U_1, \ldots, U_k \overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $W \sim \mathcal{G}(a, 1)$ independently of U's, where a is a positive integer. Further, let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$ be a vector of distinct positive integers that are all different than 1. Then, the pdf of $S = \sum_{i=1}^{k} U_i/\theta_i + W$ is given by*

$$f_S^{(k)}(s; \boldsymbol{\theta}, a) = \sum_{j=1}^{k} \Bigg(\prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{\theta_i}{\theta_i - \theta_j} \Bigg) \frac{\theta_j e^{-s\theta_j}}{(1 - \theta_j)^a} P(a, s(1 - \theta_j)), \quad s > 0, \tag{4}$$

and the corresponding distribution function is given by

$$F_S^{(k)}(s; \boldsymbol{\theta}, a) = \sum_{j=1}^{k} \Bigg(\prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{\theta_i}{\theta_i - \theta_j} \Bigg) \Bigg\{ P(a, s) - \frac{P(a, s(1 - \theta_j))}{(1 - \theta_j)^a} e^{-s\theta_j} \Bigg\}, \quad s > 0,$$

where $P(a, x) = \Gamma(a)^{-1} \int_0^x t^{a-1} e^{-t} \mathrm{d}t$ is the regularized lower incomplete gamma function.

Proof. Consider first the case $k = 1$. Then, $U_1 = \theta_1(S - W)$ and $\mathrm{d}U_1/\mathrm{d}S = \theta_1$, and so the joint pdf of (S, W) is

$$f_{S,W}^{(1)}(s, w; \theta_1, a) = \theta_1 e^{-s\theta_1} \frac{w^{a-1} e^{-w(1-\theta_1)}}{\Gamma(a)}, \quad s > w > 0.$$

Thus, the marginal pdf of S is

$$\begin{aligned} f_S^{(1)}(s; \theta_1, a) =& \theta_1 e^{-s\theta_1} \int_{w=0}^{s} \frac{w^{a-1} e^{-w(1-\theta_1)}}{\Gamma(a)} \mathrm{d}w \\ =& \frac{\theta_1 e^{-s\theta_1}}{(1 - \theta_1)^a} \int_{v=0}^{s(1-\theta_1)} \frac{v^{a-1} e^{-v}}{\Gamma(a)} \mathrm{d}v \\ =& \frac{\theta_1 e^{-s\theta_1}}{(1 - \theta_1)^a} P(a, s(1 - \theta_1)), \quad s > 0. \end{aligned}$$

On the other hand,

$$\begin{aligned}
F_S^{(1)}(s;\theta_1,a) &= \int_{t=0}^{s}\int_{w=0}^{t}\theta_1 e^{-t\theta_1}\frac{w^{a-1}e^{-w(1-\theta_1)}}{\Gamma(a)}\,\mathrm{d}w\mathrm{d}t\\
&= \int_{w=0}^{s}\int_{t=w}^{s}\theta_1 e^{-t\theta_1}\frac{w^{a-1}e^{-w(1-\theta_1)}}{\Gamma(a)}\,\mathrm{d}t\mathrm{d}w\\
&= \int_{w=0}^{s}\{e^{-w\theta_1}-e^{-s\theta_1}\}\frac{w^{a-1}e^{-w(1-\theta_1)}}{\Gamma(a)}\,\mathrm{d}w\\
&= P(a,s)-\frac{P(a,s(1-\theta_1))}{(1-\theta_1)^a}\,e^{-s\theta_1},\quad s>0.
\end{aligned}$$

Now, let $k>1$. Since $S=U+W$, $U=S-W$ and $\mathrm{d}U/\mathrm{d}S=1$, by using Lemma 1, we obtain the joint pdf of (S,W) to be

$$\begin{aligned}
f_{S,W}^{(k)}(s,w;\boldsymbol{\theta},a) &= \sum_{j=1}^{k}\left(\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{\theta_i}{\theta_i-\theta_j}\right)\theta_j e^{-(s-w)\theta_j}\frac{w^{a-1}e^{-w}}{\Gamma(a)},\quad s>w>0,\\
&= \sum_{j=1}^{k}\left(\prod_{\substack{i=1\\ i\neq j}}^{k}\frac{\theta_i}{\theta_i-\theta_j}\right)f_{S,W}^{(1)}(s,w;\theta_j,a).
\end{aligned}$$

Upon integrating with respect to w, we obtain the required result. The expression for $F_S^{(k)}(s;\boldsymbol{\theta},a)$ can be derived in an analogous manner. $\square$

Theorem 2. *Let $U_1,\ldots,U_\ell,Z_1,\ldots,Z_k,W$ be independent random variables, U's, Z's $\overset{\text{iid}}{\sim}\mathcal{E}(1)$, and $W\sim\mathcal{G}(a,1)$, $a>0$. Further, let $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_\ell)$, $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_k)$ and $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_k)$ be vectors of positive numbers such that all θ's are distinct and λ_i/μ_i is strictly increasing in i. Then, the pdf of the random variable*

$$Y=\frac{\sum_{i=1}^{\ell}U_i/\theta_i+\sum_{i=1}^{k}\lambda_i Z_i}{\sum_{i=1}^{k}\mu_i Z_i+W} \tag{5}$$

is

$$\begin{aligned}
&f_T^{(1,k)}(y;\theta_1,\boldsymbol{\lambda},\boldsymbol{\mu},a)\\
&= \frac{\theta_1}{(1+y\theta_1)^a\prod_{i=1}^{k}\{1+\theta_1(y\mu_i-\lambda_i)\}}\left\{\frac{a}{1+y\theta_1}+\sum_{i=1}^{k}\frac{\mu_i}{1+\theta_1(y\mu_i-\lambda_i)}\right\}I_{(0,\infty)}(y)\\
&-\sum_{j=1}^{k}\frac{\theta_1(\lambda_j-y\mu_j)^{a+k-1}}{(\lambda_j-y\mu_j+y)^a\{1+\theta_1(y\mu_j-\lambda_j)\}\prod_{\substack{i=1\\ i\neq j}}^{k}\{\lambda_j-\lambda_i-y(\mu_j-\mu_i)\}}\\
&\qquad\times\left\{\frac{a\lambda_j}{\lambda_j-y\mu_j+y}+\frac{\mu_j}{1+\theta_1(y\mu_j-\lambda_j)}+\sum_{\substack{i=1\\ i\neq j}}^{k}\frac{\lambda_j\mu_i-\lambda_i\mu_j}{\lambda_j-\lambda_i-y(\mu_j-\mu_i)}\right\}I_{\left(0,\frac{\lambda_j}{\mu_j}\right]}(y)
\end{aligned}$$

when $\ell = 1$, *and*

$$f_T^{(\ell,k)}(y;\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a) = \sum_{j=1}^{\ell}\left(\prod_{\substack{i=1\\ i\neq j}}^{\ell}\frac{\theta_i}{\theta_i-\theta_j}\right) f_T^{(1,k)}(y;\theta_j,\boldsymbol{\lambda},\boldsymbol{\mu},a)$$

when $\ell > 1$.

Proof. Let us first consider the case $\ell = 1$ and write for convenience U, θ instead of U_1, θ_1. The joint pdf of $U, Z_1, \ldots, Z_k, W$ is

$$\frac{w^{a-1}}{\Gamma(a)}\, e^{-u-w-\sum_{i=1}^k z_i}, \quad u, w, z_1, \ldots, z_k > 0.$$

By solving (5) (for $\ell = 1$) with respect to U, we get $U = \theta\{WY + \sum_{i=1}^k(\mu_i Y - \lambda_i)Z_i\}$ and $\mathrm{d}U/\mathrm{d}Y = \theta(W + \sum_{i=1}^k \mu_i Z_i) > 0$. So, the joint density of $Y, Z_1, \ldots, Z_k, W$ is

$$h^{(1)}(y, w, z_1, \ldots, z_k; \theta, \boldsymbol{\lambda}, \boldsymbol{\mu}, a) = \frac{\theta w^{a-1}}{\Gamma(a)}\left(w + \sum_{i=1}^k \mu_i z_i\right) e^{-(1+\theta y)w - \sum_{i=1}^k\{1+\theta(y\mu_i-\lambda_i)\}z_i},$$
$$w, z_1, \ldots, z_k, wy + \sum_{i=1}^k (y\mu_i - \lambda_i)z_i > 0. \quad (6)$$

Since $y\mu_i - \lambda_i > 0 \Leftrightarrow y > \lambda_i/\mu_i$, it follows that, for $y > \lambda_k/\mu_k$, we must integrate $h^{(1)}(y, w, z_1, \ldots, z_k)$ for all $w, z_1, \ldots, z_k > 0$. On the other hand, if $y \in (\lambda_{k-1}/\mu_{k-1}, \lambda_k/\mu_k)$, we have $y\mu_k - \lambda_k < 0$ and $y\mu_i - \lambda_i > 0$, $i = 1, \ldots, k-1$, which means that g must be first integrated for $0 < z_k < \frac{yw}{\lambda_k - y\mu_k} + \sum_{i=1}^{k-1}\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}z_i$ and then for $w, z_1, \ldots, z_{k-1} > 0$. In general, when $y \in (\lambda_{j-1}/\mu_{j-1}, \lambda_j/\mu_j)$, g must be integrated in turn for $0 < z_k < \frac{yw}{\lambda_k-y\mu_k} + \sum_{i=1}^{k-1}\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}z_i, \ldots, 0 < z_j < \frac{yw}{\lambda_j-y\mu_j} + \sum_{i=1}^{j-1}\frac{y\mu_i-\lambda_i}{\lambda_j-y\mu_j}z_i$ and $w, z_1, \ldots, z_{j-1} > 0$.

In what follows, we make use of the formula

$$\int_{x=0}^{M} e^{-\varepsilon x}(\gamma + \delta x)\mathrm{d}x = \frac{1}{\varepsilon}\left\{\gamma + \frac{\delta}{\varepsilon} - e^{-\varepsilon M}\left(\gamma + \frac{\delta}{\varepsilon} + \delta M\right)\right\}$$

which holds for any $\gamma, \delta \in \mathbb{R}$ and either $M \in [0, \infty)$, $\varepsilon \in \mathbb{R}$ (where in the case $\varepsilon = 0$, we must take the limit of the right hand side as $\varepsilon \to 0$), or $M = \infty$, $\varepsilon > 0$. Moreover, in order to avoid unnecessary complicated evaluations, we shall derive the density for all y's outside the set

$$A = \{y:\ 1 + \theta(y\mu_j - \lambda_j) = 0,\ \text{for some } j = 1, \ldots, k\}$$

which is finite and hence has zero probability.

Let us first consider $y > \lambda_k/\mu_k$. Then, the density of Y at y is

$$\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_k=0}^{\infty}\frac{\theta w^{a-1}}{\Gamma(a)}\left(w + \sum_{i=1}^k \mu_i z_i\right) e^{-(1+\theta y)w - \sum_{i=1}^k\{1+\theta(y\mu_i-\lambda_i)\}z_i}\mathrm{d}z_k\cdots\mathrm{d}z_1\mathrm{d}w$$

$$
\begin{aligned}
=&\frac{\theta}{1+\theta(y\mu_k-\lambda_k)}\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-1}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}\,e^{-(1+\theta y)w-\sum\limits_{i=1}^{k-1}\{1+\theta(y\mu_i-\lambda_i)\}z_i}\\
&\qquad\times\left\{w+\sum_{i=1}^{k-1}\mu_i z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}\right\}\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1\mathrm{d}w \qquad (7)\\
=&\frac{\theta}{\prod\limits_{i=k-1}^{k}\{1+\theta(y\mu_i-\lambda_i)\}}\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-2}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}e^{-(1+\theta y)w-\sum\limits_{i=1}^{k-2}\{1+\theta(y\mu_i-\lambda_i)\}z_i}\\
&\qquad\times\left\{w+\sum_{i=1}^{k-2}\mu_i z_i+\sum_{i=k-1}^{k}\frac{\mu_i}{1+\theta(y\mu_i-\lambda_i)}\right\}\mathrm{d}z_{k-2}\cdots\mathrm{d}z_1\mathrm{d}w\\
=&\cdots\cdots\\
=&\frac{\theta}{\prod\limits_{i=1}^{k}\{1+\theta(y\mu_i-\lambda_i)\}}\int_{w=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}\,e^{-(1+\theta y)w}\left\{w+\sum_{i=1}^{k}\frac{\mu_i}{1+\theta(y\mu_i-\lambda_i)}\right\}\mathrm{d}w\\
=&\frac{\theta}{(1+\theta y)^a\prod\limits_{i=1}^{k}\{1+\theta(y\mu_i-\lambda_i)\}}\left\{\frac{a}{1+\theta y}+\sum_{i=1}^{k}\frac{\mu_i}{1+\theta(y\mu_i-\lambda_i)}\right\} \qquad (8)
\end{aligned}
$$

as stated. Next, for $y\in(\lambda_{k-1}/\mu_{k-1},\lambda_k/\mu_k)\cap A^{\complement}$, the density of Y is

$$
\begin{aligned}
&\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-1}=0}^{\infty}\int_{z_k=0}^{\frac{yw}{\lambda_k-y\mu_k}+\sum\limits_{i=1}^{k-1}\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}z_i}\frac{\theta w^{a-1}}{\Gamma(a)}e^{-(1+\theta y)w-\sum\limits_{i=1}^{k}\{1+\theta(y\mu_i-\lambda_i)\}z_i}\\
&\qquad\times\left(w+\sum_{i=1}^{k}\mu_i z_i\right)\mathrm{d}z_k\cdots\mathrm{d}z_1\mathrm{d}w\\
=&\frac{\theta}{1+\theta(y\mu_k-\lambda_k)}\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-1}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}e^{-(1+\theta y)w-\sum\limits_{i=1}^{k-1}\{1+\theta(y\mu_i-\lambda_i)\}z_i}\\
&\times\left\{w+\sum_{i=1}^{k-1}\mu_i z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}-e^{-\{1+\theta(y\mu_k-\lambda_k)\}\left(\frac{yw}{\lambda_k-y\mu_k}+\sum\limits_{i=1}^{k-1}\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}z_i\right)}\right.\\
&\left.\times\left[w+\sum_{i=1}^{k-1}\mu_i z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}+\mu_k\left(\frac{yw}{\lambda_k-y\mu_k}+\sum_{i=1}^{k-1}\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}z_i\right)\right]\right\}\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1\mathrm{d}w\\
=&\frac{\theta}{1+\theta(y\mu_k-\lambda_k)}\left\{\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-1}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}e^{-(1+\theta y)w-\sum\limits_{i=1}^{k-1}\{1+\theta(y\mu_i-\lambda_i)\}z_i}\right.\\
&\qquad\times\left(w+\sum_{i=1}^{k-1}\mu_i z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}\right)\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1\mathrm{d}w\\
&\quad-\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-1}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}\,e^{-\left(1+\frac{y}{\lambda_k-y\mu_k}\right)w-\sum\limits_{i=1}^{k-1}\left(1+\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}\right)z_i}\\
&\quad\left.\times\left(\frac{\lambda_k w}{\lambda_k-y\mu_k}+\sum_{i=1}^{k-1}\frac{\lambda_k\mu_i-\lambda_i\mu_k}{\lambda_k-y\mu_k}z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}\right)\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1\mathrm{d}w\right\}. \qquad (9)
\end{aligned}
$$

Note that the first k-fold integral is the same as that in (7) and so it is equal to (8). On the other hand, the second k-fold integral can be evaluated similarly and can be shown to equal

$$\frac{\theta(\lambda_k - y\mu_k)^{a+k-1}}{(\lambda_k - y\mu_k + y)^a\{1+\theta(y\mu_k - \lambda_k)\}\prod_{i=1}^{k-1}\{\lambda_k - \lambda_i - y(\mu_k - \mu_i)\}}$$
$$\times\left\{\frac{a\lambda_k}{\lambda_k - y\mu_k + y} + \frac{\mu_k}{1+\theta(y\mu_k - \lambda_k)} + \sum_{i=1}^{k-1}\frac{\lambda_k\mu_i - \lambda_i\mu_k}{\lambda_k - \lambda_i - y(\mu_k - \mu_i)}\right\}. \quad (10)$$

This proves the assertion for $y \in (\lambda_{k-1}/\mu_{k-1}, \lambda_k/\mu_k) \cap A^{\complement}$. In order to prove it for all of the remaining intervals $(\lambda_{j-1}/\mu_{j-1}, \lambda_j/\mu_j) \cap A^{\complement}$, we can use induction. However, here we will go just one step further and prove the result for $y \in (\lambda_{k-2}/\mu_{k-2}, \lambda_{k-1}/\mu_{k-1}) \cap A^{\complement}$. In this case, the density of Y can be found by continuing from (9) and integrating over z_{k-1} from 0 to $\frac{yw}{\lambda_{k-1}-y\mu_{k-1}} + \sum_{i=1}^{k-2}\frac{y\mu_i-\lambda_i}{\lambda_{k-1}-y\mu_{k-1}}z_i$ instead of 0 to ∞. After the integration with respect to z_{k-1}, the integral becomes

$$\frac{\theta}{1+\theta(y\mu_k-\lambda_k)}\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-2}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}$$
$$\times\Bigg\{\frac{1}{1+\theta(y\mu_{k-1}-\lambda_{k-1})}e^{-(1+\theta y)w-\sum_{i=1}^{k-2}\{1+\theta(y\mu_i-\lambda_i)\}z_i}$$
$$\times\Bigg[w+\sum_{i=1}^{k-2}\mu_i z_i + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} + \frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})}$$
$$- e^{-\{1+\theta(y\mu_{k-1}-\lambda_{k-1})\}\left(\frac{yw}{\lambda_{k-1}-y\mu_{k-1}}+\sum_{i=1}^{k-2}\frac{y\mu_i-\lambda_i}{\lambda_{k-1}-y\mu_{k-1}}z_i\right)}\Bigg(w+\sum_{i=1}^{k-2}\mu_i z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}$$
$$+\frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})}+\mu_{k-1}\left\{\frac{yw}{\lambda_{k-1}-y\mu_{k-1}}+\sum_{i=1}^{k-2}\frac{y\mu_i-\lambda_i}{\lambda_{k-1}-y\mu_{k-1}}z_i\right\}\Bigg)\Bigg]$$
$$-\frac{\lambda_k-y\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})}e^{-\left(1+\frac{y}{\lambda_k-y\mu_k}\right)w-\sum_{i=1}^{k-2}\left(1+\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}\right)z_i}$$
$$\times\Bigg[\frac{\lambda_k w}{\lambda_k-y\mu_k}+\sum_{i=1}^{k-2}\frac{\lambda_k\mu_i-\lambda_i\mu_k}{\lambda_k-y\mu_k}z_i+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}+\frac{\lambda_k\mu_{k-1}-\lambda_{k-1}\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})}$$
$$-e^{-\left(1+\frac{y\mu_{k-1}-\lambda_{k-1}}{\lambda_k-y\mu_k}\right)\left(\frac{yw}{\lambda_{k-1}-y\mu_{k-1}}+\sum_{i=1}^{k-2}\frac{y\mu_i-\lambda_i}{\lambda_{k-1}-y\mu_{k-1}}z_i\right)}\Bigg(\frac{\lambda_k w}{\lambda_k-y\mu_k}+\sum_{i=1}^{k-2}\frac{\lambda_k\mu_i-\lambda_i\mu_k}{\lambda_k-y\mu_k}z_i$$
$$+\frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)}+\frac{\lambda_k\mu_{k-1}-\lambda_{k-1}\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})}$$
$$+\frac{\lambda_k\mu_{k-1}-\lambda_{k-1}\mu_k}{\lambda_k-y\mu_k}\left\{\frac{\lambda_k w}{\lambda_k-y\mu_k}+\sum_{i=1}^{k-2}\frac{\lambda_k\mu_i-\lambda_i\mu_k}{\lambda_k-y\mu_k}z_i\right\}\Bigg)\Bigg]\Bigg\}\mathrm{d}z_{k-2}\cdots\mathrm{d}z_1\mathrm{d}w$$
$$=\Bigg\{\frac{\theta}{\prod_{i=k-1}^{k}\{1+\theta(y\mu_i-\lambda_i)\}}\int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-2}=0}^{\infty}\frac{w^{a-1}}{\Gamma(a)}e^{-(1+\theta y)w-\sum_{i=1}^{k-2}\{1+\theta(y\mu_i-\lambda_i)\}z_i}$$

$$\times \left(w + \sum_{i=1}^{k-2} \mu_i z_i + \frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right) \mathrm{d}z_{k-2}\cdots \mathrm{d}z_1 \mathrm{d}w \Bigg\}$$

$$- \Bigg\{ \frac{\theta(\lambda_k - y\mu_k)}{\{1+\theta(y\mu_k-\lambda_k)\}\{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})\}} \int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-2}=0}^{\infty} \frac{w^{a-1}}{\Gamma(a)}$$

$$\times e^{-\left(1+\frac{y}{\lambda_k-y\mu_k}\right)w - \sum_{i=1}^{k-2}\left(1+\frac{y\mu_i-\lambda_i}{\lambda_k-y\mu_k}\right)z_i} \left(\frac{\lambda_k w}{\lambda_k - y\mu_k} + \sum_{i=1}^{k-2} \frac{\lambda_k\mu_i - \lambda_i\mu_k}{\lambda_k - y\mu_k} z_i \right.$$

$$\left. + \frac{\lambda_k\mu_{k-1}-\lambda_{k-1}\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right) \mathrm{d}z_{k-2}\cdots \mathrm{d}z_1 \mathrm{d}w \Bigg\}$$

$$- \Bigg\{ \frac{\theta}{1+\theta(y\mu_k-\lambda_k)} \int_{w=0}^{\infty}\int_{z_1=0}^{\infty}\cdots\int_{z_{k-2}=0}^{\infty} \frac{w^{a-1}}{\Gamma(a)} e^{-\left(1+\frac{y}{\lambda_{k-1}-y\mu_{k-1}}\right)w - \sum_{i=1}^{k-2}\left(1+\frac{y\mu_i-\lambda_i}{\lambda_{k-1}-y\mu_{k-1}}\right)z_i}$$

$$\times \left[\frac{1}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} \left(w + \sum_{i=1}^{k-2}\mu_i z_i + \frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right) \right.$$

$$- \frac{\lambda_k - y\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})} \left(\frac{\lambda_{k-1}w}{\lambda_{k-1}-y\mu_{k-1}} + \sum_{i=1}^{k-2} \frac{\lambda_{k-1}\mu_i - \lambda_i\mu_{k-1}}{\lambda_{k-1}-y\mu_{k-1}} z_i \right.$$

$$\left.\left. + \frac{\lambda_{k-1}\mu_k - \lambda_k\mu_{k-1}}{\lambda_{k-1}-\lambda_k-y(\mu_{k-1}-\mu_k)} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right) \right] \mathrm{d}z_{k-2}\cdots \mathrm{d}z_1 \mathrm{d}w \Bigg\}. \tag{11}$$

Here, it can be seen that after performing all necessary integrations, the terms in the first two braces reduce to (8) and (10), respectively. On the other hand, by using the facts that

$$\frac{1}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} - \frac{\lambda_k - y\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})}$$
$$= \frac{\{1+\theta(y\mu_k-\lambda_k)\}(\lambda_{k-1}-y\mu_{k-1})}{\{1+\theta(y\mu_{k-1}-\lambda_{k-1})\}\{\lambda_{k-1}-\lambda_k-y(\mu_{k-1}-\mu_k)\}}$$

and

$$\frac{1}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} \left(\frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right)$$
$$- \frac{\lambda_k - y\mu_k}{\lambda_k-\lambda_{k-1}-y(\mu_k-\mu_{k-1})} \left(\frac{\lambda_{k-1}\mu_k-\lambda_k\mu_{k-1}}{\lambda_{k-1}-\lambda_k-y(\mu_{k-1}-\mu_k)} + \frac{\mu_k}{1+\theta(y\mu_k-\lambda_k)} \right)$$
$$= \frac{\{1+\theta(y\mu_k-\lambda_k)\}(\lambda_{k-1}-y\mu_{k-1})}{\{1+\theta(y\mu_{k-1}-\lambda_{k-1})\}\{\lambda_{k-1}-\lambda_k-y(\mu_{k-1}-\mu_k)\}}$$
$$\times \left(\frac{\lambda_{k-1}\mu_k-\lambda_k\mu_{k-1}}{\lambda_{k-1}-\lambda_k-y(\mu_{k-1}-\mu_k)} + \frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} \right)$$

we obtain from (11), after carrying out all the integrations,

$$\frac{\theta(\lambda_{k-1}-y\mu_{k-1})^{a+k-1}}{(\lambda_{k-1}-y\mu_{k-1}+y)^a\{1+\theta(y\mu_{k-1}-\lambda_{k-1})\} \prod_{\substack{i=1\\ i\neq k-1}}^{k} \{\lambda_{k-1}-\lambda_i-y(\mu_{k-1}-\mu_i)\}}$$

$$\times \left\{ \frac{a\lambda_{k-1}}{\lambda_{k-1}-y\mu_{k-1}+y} + \frac{\mu_{k-1}}{1+\theta(y\mu_{k-1}-\lambda_{k-1})} + \sum_{\substack{i=1\\ i\neq k-1}}^{k} \frac{\lambda_{k-1}\mu_i-\lambda_i\mu_{k-1}}{\lambda_{k-1}-\lambda_i-y(\mu_{k-1}-\mu_i)} \right\}$$

and this establishes the result for $\ell = 1$.

In order to prove it for $\ell > 1$, set $U = \sum_{i=1}^{\ell} U_i/\theta_i$ so that we have

$$Y = \frac{U + \sum_{i=1}^{k} \lambda_i Z_i}{\sum_{i=1}^{k} \mu_i Z_i + W}.$$

By solving for U, we get $U = WY + \sum_{i=1}^{k}(Y\mu_i - \lambda_i)Z_i$ and $\mathrm{d}U/\mathrm{d}Y = W + \sum_{i=1}^{k} \mu_i Z_i$, and upon using Lemma 1, we conclude that the joint pdf of Y, Z's, and W is

$$\begin{aligned} h^{(\ell)}&(y, w, z_1, \ldots, z_k; \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a) \\ &= \sum_{j=1}^{\ell} \left(\prod_{\substack{i=1 \\ i \neq j}}^{\ell} \frac{\theta_i}{\theta_i - \theta_j} \right) \left\{ \frac{\theta_j w^{a-1}}{\Gamma(a)} \left(w + \sum_{i=1}^{k} \mu_i z_i \right) e^{-(1+y\theta_j)w - \sum_{i=1}^{k}\{1+\theta_j(y\mu_i - \lambda_i)z_i} \right\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad w, z_1, \ldots, z_k, wy + \sum_{i=1}^{k}(y\mu_i - \lambda_i)z_i > 0. \end{aligned}$$

Now, in order to integrate out z's and w, we have to work with terms within each brace separately. However, all these quantities have the form of (6), and this yields the result. $\square$

Remark 1. The distribution presented in Theorem 2 as well as those derived in Theorems 3 and 4 below may possibly be deduced by the work of Provost and Rudiuk (1994). These authors discussed the distribution of the ratio of dependent linear combinations of chi-square random variables (which are in fact exponential random variables when the degrees of freedom equal two) via inverse Mellin transforms. However, in our special case, we have chosen to derive the required distributions in a straightforward manner through standard transformations of random variables rather than to try to invert the corresponding Mellin transforms which are expressed as infinite power series.

Theorem 3. *Let $Z_1, \ldots, Z_k, W$ be independent random variables, Z's $\overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $W \sim \mathcal{G}(a, 1)$, $a > 0$. Further, let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k)$ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_k)$ be vectors of positive numbers such that λ_i/μ_i is strictly increasing in i. Then, the pdf of the random variable*

$$Y = \frac{\sum_{i=1}^{k} \lambda_i Z_i}{\sum_{i=1}^{k} \mu_i Z_i + W} \tag{12}$$

is

$$\begin{aligned} f_T^{(0,k)}(y; \boldsymbol{\lambda}, \boldsymbol{\mu}, a) = \sum_{j=1}^{k} &\frac{(\lambda_j - y\mu_j)^{a+k-2}}{(\lambda_j - y\mu_j + y)^a \prod_{\substack{i=1 \\ i \neq j}}^{k} \{\lambda_j - \lambda_i - y(\mu_j - \mu_i)\}} \\ &\times \left\{ \frac{a\lambda_j}{\lambda_j - y\mu_j + y} + \sum_{\substack{i=1 \\ i \neq j}}^{k} \frac{\lambda_j \mu_i - \lambda_i \mu_j}{\lambda_j - \lambda_i - y(\mu_j - \mu_i)} \right\} I_{\left(0, \frac{\lambda_j}{\mu_j}\right]}(y). \end{aligned} \tag{13}$$

Proof. From (12), we get $Z_k = \sum_{i=1}^{k-1} \frac{Y\mu_i - \lambda_i}{\lambda_k - Y\mu_k} Z_i + \frac{YW}{\lambda_k - Y\mu_k}$ with $\mathrm{d}Z_k/\mathrm{d}Y = \sum_{i=1}^{k-1} \frac{\lambda_k\mu_i - \lambda_i\mu_k}{(\lambda_k - Y\mu_k)^2} Z_i + \frac{W\lambda_k}{(\lambda_k - Y\mu_k)^2}$, and so the joint pdf of $Z_1, \ldots, Z_{k-1}, W, Y$ is

$$h(y, w, z_1, \ldots, z_{k-1}; \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$$
$$= \frac{w^{a-1}}{\Gamma(a)(\lambda_k - y\mu_k)^2} \left\{ w\lambda_k + \sum_{i=1}^{k-1} (\lambda_k\mu_i - \lambda_i\mu_k) z_i \right\} e^{-\left(1+\frac{y}{\lambda_k - y\mu_k}\right)w - \sum_{i=1}^{k-1} \left(1+\frac{y\mu_i - \lambda_i}{\lambda_k - y\mu_k}\right) z_i},$$
$$w, z_1, \ldots, z_{k-1}, \frac{wy}{\lambda_k - y\mu_k} + \sum_{i=1}^{k-1} \frac{y\mu_i - \lambda_i}{\lambda_k - y\mu_k} z_i > 0.$$

From the last inequality, we conclude that if $y > \lambda_k/\mu_k$ then the above joint pdf equals zero. Now, working as in the proof of Theorem 2, one has to consider here the cases $y \in (\lambda_{k-1}/\mu_{k-1}, \lambda_k/\mu_k)$, $(\lambda_{k-2}/\mu_{k-2}, \lambda_{k-1}/\mu_{k-1})$, …, $(0, \lambda_1/\mu_1)$ and carry out the integrations to arrive at the required result. □

Remark 2. Let $U \sim \mathcal{E}(1)$ independently of Z's and W. Then, by Theorem 2, for any $\theta > 0$,

$$Y'_\theta = \frac{U/\theta + \sum_{i=1}^k \lambda_i Z_i}{\sum_{i=1}^k \mu_i Z_i + W} \sim f_T^{(1,k)}(y; \theta, \boldsymbol{\lambda}, \boldsymbol{\mu}, a).$$

We have $\lim_{\theta\to\infty} Y'_\theta = Y$ almost surely and consequently in distribution as well. On the other hand, it can be verified that $\lim_{\theta\to\infty} f_T^{(1,k)}(y; \theta, \boldsymbol{\lambda}, \boldsymbol{\mu}, a) = f_T^{(0,k)}(y; \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ for all $y > 0$. Hence, Theorem 3 could be deduced from Theorem 2 by using a limiting argument provided some additional regularity conditions would be satisfied. But, direct transformation of variables is sufficient for proving the result.

Theorem 4. *Let $U_1, \ldots, U_\ell, Z_1, \ldots, Z_k \overset{\text{iid}}{\sim} \mathcal{E}(1)$, and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_\ell)$, $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_k)$ and $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_k)$ be vectors of positive numbers such that all θ's are distinct and λ_i/μ_i is strictly increasing in i. Then,*

$$Y = \frac{\sum_{i=1}^\ell U_i/\theta_i + \sum_{i=1}^k \lambda_i Z_i}{\sum_{i=1}^k \mu_i Z_i} \sim f_T^{(\ell,k)}(y; \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, 0).$$

Proof. The proof proceeds exactly as that of Theorem 2, and everything works in the same way except that there is no integral here with respect to w. □

Theorem 5. *Let $U_1, \ldots, U_\ell, W$ be independent random variables, U's $\overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $W \sim \mathcal{G}(a, 1)$, $a > 0$. Further, let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_\ell)$ be a vector of distinct positive numbers. Then, the pdf of the random variable*

$$Y = \frac{\sum_{i=1}^\ell U_i/\theta_i}{W}$$

is

$$f_T^{(\ell,0)}(y; \boldsymbol{\theta}, a) = \sum_{j=1}^\ell \left(\prod_{\substack{i=1 \\ i\neq j}}^\ell \frac{\theta_i}{\theta_i - \theta_j} \right) \frac{a\theta_j}{(1 + y\theta_j)^{a+1}}, \quad y > 0.$$

Proof. Using Lemma 1, it can be shown rather easily. $\square$

Theorem 6. *Let U, Z, W be independent random variables, $U, Z \overset{\text{iid}}{\sim} \mathcal{E}(1)$ and $W \sim \mathcal{G}(a, 1)$, $a > 0$. Also, let $c > 0$. Then, the pdf of the random variable*

$$Y = \frac{c(U - Z)}{U + Z + W}$$

is given by

$$g_T(y; c, a) = \frac{a+1}{2c}\left(1 - \frac{|y|}{c}\right)^a I_{(-c,c)}(y).$$

Proof. Since c is just a scale parameter, consider first $c = 1$ and set $R = U - Z$, $S = U + Z$. Then, $U = (S + R)/2$, $Z = (S - R)/2$, with Jacobian $1/2$ and so the joint density of R, S, W is

$$g_{R,S,W}(r, s, w) = \frac{w^{a-1}}{2\Gamma(a)}\, e^{-s-w}, \quad w > 0, s + r > 0, s - r > 0.$$

Since $Y = R/(S + W)$, we have $R = Y(S + W)$, $\mathrm{d}R/\mathrm{d}Y = S + W$, and the joint density of Y, S, W is

$$g_{T,S,W}(y, s, w) = \frac{w^{a-1}}{2\Gamma(a)}\, e^{-s-w}, \quad w > 0, (1 + y)s + yw > 0, (1 - y)s - yw > 0.$$

But

$$\begin{aligned}
\{(y, s, w) : w > 0, &(1 + y)s + yw > 0, (1 - y)s - yw > 0\} \\
&= \{(y, s, w) : w > 0, -1 < y < 0, s > -yw/(1 + y)\} \cup \\
&\quad\ \{(y, s, w) : w > 0, 0 \leqslant y < 1, s > yw/(1 - y)\}
\end{aligned}$$

and so,

$$\begin{aligned}
g_T(y; c = 1, a) &= \begin{cases} \int_{w=0}^{\infty} \int_{s=-yw/(1+y)}^{\infty} \frac{w^{a-1}}{2\Gamma(a)}\, e^{-s-w} \mathrm{d}s\mathrm{d}w, & -1 < y < 0, \\ \int_{w=0}^{\infty} \int_{s=yw/(1-y)}^{\infty} \frac{w^{a-1}}{2\Gamma(a)}\, e^{-s-w} \mathrm{d}s\mathrm{d}w, & 0 \leqslant y < 1 \end{cases} \\
&= \frac{a+1}{2}(1 - |y|)^a I_{(-1,1)}(y).
\end{aligned}$$

$\square$

References

Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (2008). *A First Course in Order Statistics*, Classic Edition, SIAM, Philadelphia.

Bain, L.J. and Engelhardt, M. (1973). Interval estimation for the two-parameter double exponential distribution, *Technometrics*, **15**, 875–887.

Balakrishnan, N. (2007). Progressive censoring methodology: An appraisal (with discussions), *Test*, **16**, 211-296.

Balakrishnan, N. and Cutler, C.D. (1995). Maximum likelihood estimation of Laplace parameters based on Type-II censored samples, In *Statistical Theory and Applications: Papers in Honor of Herbert A. David* (Eds., H.N. Nagaraja, P.K. Sen and D.F. Morrison), pp. 145–151, Springer-Verlag, New York.

Chernoff, H., Gastwirth, J.L. and Johns, M.V. (1967). Asympotic distribution of linear combinations of functions of order statistics with applications to estimation, *The Annals of Mathematical Statistics*, **38**, 52–72.

Childs, A. and Balakrishnan, N. (1996). Conditional inference procedures for the Laplace distribution based on type-II right censored samples, *Statistics & Probability Letters*, **31**, 31–39.

Childs, A. and Balakrishnan, N. (1997). Maximum likelihood estimation of Laplace parameters based on general Type-II censored samples, *Statistical Papers*, **38**, 343–349.

David, H.A. and Nagaraja, H.N. (2003). *Order Statistics*, Third Edition, John Wiley & Sons, Hoboken, New Jersey.

Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*, Chapman & Hall, New York.

Grice, J.V., Bain, L.J. and Engelhardt, M. (1978). Comparison of conditional and unconditional confidence intervals for the double exponential distribution, *Communications in Statistics – Simulation and Computation*, **7**, 515–524.

Iliopoulos, G. and Balakrishnan, N. (2009). Conditional independence of blocked ordered data, *Statistics & Probability Letters*, **79**, 1008–1015.

Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions, Vol. 2*, Second edition, John Wiley & Sons, New York.

Kappenman, R.F. (1975). Conditional confidence intervals for double exponential distribution parameters, *Technometrics*, **17**, 233–235.

Kotz, S., Kozubowski, T.J. and Podgórski, K. (2001). *The Laplace Distribution and Generalizations*, Birkhäuser, Boston.

Provost, S.B. and Rudiuk, E.M. (1994). The exact density function of the ratio of two dependent linear combinations of chi-square variables, *Annals of the Institute of Statistical Mathematics*, **46**, 557–571.

Sansing, R.C. (1976). The t-statistic for a double exponential distribution, *SIAM Journal on Applied Mathematics*, **31**, 634–645.

n	90%	95%	99%
15	84.8%	91.6%	98.4%
20	85.9%	92.2%	98.3%
25	85.6%	91.8%	98.0%
30	86.4%	92.3%	98.1%
35	86.1%	92.1%	98.0%

n	90%	95%	99%
40	86.7%	92.5%	98.1%
45	86.5%	92.3%	98.0%
50	86.9%	92.7%	98.1%
55	86.8%	92.5%	98.0%
60	87.1%	92.8%	98.1%

Table 1: Exact coverage probabilities of confidence intervals for μ in the case of complete samples based on the normal approximation proposed by Bain and Engelhardt (1973).

---

n	90%	95%	99%
15	87.6%	93.0%	98.3%
20	87.8%	93.6%	98.6%
25	88.3%	93.8%	98.7%
30	88.6%	94.4%	99.0%
35	88.9%	94.4%	98.8%

n	90%	95%	99%
40	89.6%	94.9%	98.9%
45	89.4%	94.7%	99.1%
50	89.9%	95.0%	99.0%
55	90.0%	95.1%	99.0%
60	89.1%	94.7%	99.0%

Table 2: Estimated coverage probabilities of bootstrap confidence intervals for μ in the case of complete samples based on 10000 simulations and 1000 bootstrap samples.

---

n	90%	95%	99%
15	89.5%	94.3%	98.0%
20	89.8%	94.4%	98.3%
25	89.3%	94.5%	98.5%
30	90.3%	94.9%	98.6%
35	90.4%	95.0%	98.8%

n	90%	95%	99%
40	90.5%	94.8%	98.6%
45	90.3%	94.9%	98.6%
50	89.8%	94.6%	98.6%
55	90.4%	95.0%	98.5%
60	90.2%	94.7%	98.7%

Table 3: Estimated coverage probabilities of bootstrap confidence intervals for σ in the case of complete samples based on 10000 simulations and 1000 bootstrap samples.

---

n	0.1	0.05	0.025	0.01	0.005
2	2.5	5	10	25	50
3	1.6646	2.4271	3.3166	5.0990	7.1414
4	1.0548	1.5024	2.0000	2.7583	3.4456
5	.9418	1.3144	1.6841	2.1910	2.6121
6	.7532	1.0366	1.3226	1.7135	2.0252
7	.7068	.9702	1.2237	1.5524	1.8030
8	.6093	.8287	1.0421	1.3214	1.5343
9	.5843	.7949	.9945	1.2483	1.4364
10	.5226	.7061	.8812	1.1058	1.2730
11	.5070	.6856	.8536	1.0647	1.2192
12	.4635	.6234	.7744	.9656	1.1062
13	.4529	.6098	.7564	.9396	1.0725
14	.4201	.5631	.6973	.8656	.9883
15	.4125	.5535	.6847	.8478	.9655
16	.3865	.5168	.6383	.7898	.8997
17	.3808	.5096	.6290	.7769	.8834
18	.3597	.4799	.5914	.7300	.8301
19	.3552	.4743	.5844	.7203	.8178
20	.3375	.4495	.5531	.6813	.7736
21	.3340	.4451	.5476	.6737	.7641
22	.3189	.4241	.5210	.6407	.7266
23	.3160	.4205	.5166	.6347	.7190
24	.3030	.4023	.4937	.6062	.6867
25	.3006	.3994	.4900	.6013	.6806
26	.2892	.3835	.4701	.5764	.6524
27	.2871	.3810	.4670	.5724	.6474
28	.2770	.3670	.4494	.5504	.6225
29	.2752	.3648	.4468	.5470	.6183
30	.2662	.3523	.4311	.5275	.5962
31	.2647	.3505	.4289	.5246	.5926
32	.2566	.3392	.4148	.5071	.5727
33	.2552	.3377	.4128	.5045	.5696
34	.2478	.3275	.4001	.4887	.5517
35	.2467	.3261	.3984	.4865	.5490
36	.2399	.3168	.3868	.4721	.5326
37	.2389	.3156	.3853	.4702	.5303
38	.2327	.3070	.3747	.4570	.5154
39	.2318	.3060	.3733	.4553	.5133
40	.2261	.2981	.3636	.4432	.5000

Table 4: Upper quantiles of $T = (\hat{\mu} - \mu)/\hat{\sigma}$ in the case of complete samples. By symmetry, the corresponding lower quantiles are their negatives.

n	0.995	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
2	0.0050	0.0100	0.0250	0.0501	0.1006	1.6359	2.0565	2.4659	2.9951	3.3887
3	0.0473	0.0672	0.1073	0.1541	0.2246	1.4088	1.6976	1.9763	2.3345	2.6000
4	0.1103	0.1408	0.1962	0.2553	0.3383	1.5006	1.7616	2.0093	2.3228	2.5522
5	0.1591	0.1930	0.2521	0.3124	0.3941	1.4188	1.6374	1.8433	2.1022	2.2908
6	0.2087	0.2459	0.3091	0.3720	0.4554	1.4333	1.6345	1.8228	2.0582	2.2288
7	0.2448	0.2827	0.3459	0.4079	0.4887	1.3884	1.5684	1.7359	1.9444	2.0950
8	0.2818	0.3209	0.3852	0.4474	0.5276	1.3883	1.5567	1.7129	1.9066	2.0460
9	0.3092	0.3482	0.4117	0.4726	0.5503	1.3590	1.5144	1.6579	1.8354	1.9628
10	0.3378	0.3771	0.4406	0.5010	0.5776	1.3551	1.5023	1.6379	1.8050	1.9248
11	0.3595	0.3984	0.4609	0.5200	0.5944	1.3342	1.4723	1.5993	1.7554	1.8671
12	0.3824	0.4112	0.4833	0.5416	0.6147	1.3294	1.4613	1.5824	1.7309	1.8369
13	0.4000	0.4384	0.4995	0.5565	0.6278	1.3134	1.4387	1.5534	1.6939	1.7941
14	0.4188	0.4570	0.5174	0.5737	0.6437	1.3086	1.4290	1.5391	1.6737	1.7695
15	0.4336	0.4713	0.5307	0.5859	0.6542	1.2959	1.4112	1.5164	1.6449	1.7362
16	0.4494	0.4867	0.5455	0.5999	0.6671	1.2914	1.4027	1.5041	1.6279	1.7157
17	0.4620	0.4989	0.5567	0.6100	0.6758	1.2810	1.3882	1.4858	1.6046	1.6889
18	0.4755	0.5120	0.5692	0.6218	0.6845	1.2767	1.3807	1.4752	1.5901	1.6716
19	0.4864	0.5224	0.5787	0.6304	0.6938	1.2681	1.3686	1.4599	1.5708	1.6494
20	0.4981	0.5338	0.5894	0.6404	0.7029	1.2642	1.3619	1.4506	1.5583	1.6345
21	0.5076	0.5429	0.5977	0.6479	0.7092	1.2567	1.3517	1.4377	1.5420	1.6157
22	0.5179	0.5528	0.6071	0.6565	0.7170	1.2532	1.3457	1.4295	1.5310	1.6027
23	0.5264	0.5609	0.6144	0.6631	0.7225	1.2467	1.3368	1.4183	1.5170	1.5867
24	0.5355	0.5697	0.6226	0.6707	0.7293	1.2434	1.3315	1.4110	1.5073	1.5752
25	0.5431	0.5769	0.6291	0.6765	0.7342	1.2378	1.3237	1.4013	1.4951	1.5612
26	0.5513	0.5867	0.6364	0.6832	0.7401	1.2348	1.3189	1.3947	1.4865	1.5511
27	0.5581	0.5912	0.6422	0.6884	0.7445	1.2298	1.3120	1.3861	1.4757	1.5388
28	0.5655	0.5983	0.6487	0.6944	0.7498	1.2270	1.3076	1.3802	1.4679	1.5296
29	0.5717	0.6041	0.6540	0.6991	0.7537	1.2225	1.3015	1.3726	1.4584	1.5187
30	0.5784	0.6105	0.6599	0.7045	0.7584	1.2199	1.2974	1.3672	1.4513	1.5105
31	0.5841	0.6159	0.6647	0.7088	0.7620	1.2159	1.2919	1.3603	1.4427	1.5007
32	0.5902	0.6217	0.6701	0.7136	0.7663	1.2135	1.2882	1.3554	1.4363	1.4932
33	0.5955	0.6266	0.6745	0.7175	0.7695	1.2099	1.2832	1.3492	1.4286	1.4844
34	0.6011	0.6320	0.6793	0.7220	0.7734	1.2076	1.2798	1.3447	1.4227	1.4775
35	0.6059	0.6365	0.6834	0.7256	0.7764	1.2043	1.2753	1.3390	1.4157	1.4695
36	0.6111	0.6414	0.6879	0.7297	0.7799	1.2022	1.2721	1.3348	1.4103	1.4632
37	0.6155	0.6456	0.6916	0.7329	0.7826	1.1991	1.2679	1.3297	1.4039	1.4559
38	0.6203	0.6502	0.6958	0.7367	0.7859	1.1972	1.2650	1.3258	1.3989	1.4501
39	0.6243	0.6541	0.6992	0.7398	0.7884	1.1944	1.2612	1.3210	1.3930	1.4433
40	0.6290	0.6583	0.7031	0.7432	0.7914	1.1925	1.2584	1.3175	1.3883	1.4380

Table 5: Quantiles of $S = \hat{\sigma}/\sigma$ in the case of complete samples.

1.96	1.96	3.60	3.80	4.79	5.66	5.76	5.78	6.27	6.30	6.76
7.65	7.84	7.99	8.51	9.18	10.13	10.24	10.25	10.43	11.45	11.48
11.75	11.81	12.34	12.78	13.06	13.29	13.98	14.18	14.40	16.22	17.06

Table 6: Data on differences in flood stages for two stations on the Fox River, Wisconsin, for 33 different years.