Exact prediction intervals for order statistics from the Laplace distribution based on the MLEs

G. Iliopoulos* S. M. T. K. MirMostafaee†

Abstract

In this work we construct exact prediction intervals for order statistics from the Laplace (double exponential) distribution. We consider both the one- and two-sample prediction cases. The intervals are based on certain pivotal quantities that employ the corresponding maximum likelihood predictors and the predictive maximum likelihood estimators of the unknown parameters. Similarly to Iliopoulos and Balakrishnan (2011), we express the distributions of the pivotal quantities as mixtures of ratios of linear combinations of independent standard exponential random variables. Since these distributions are in closed form we solve numerically the corresponding equations and obtain their exact quantiles. Tables containing selected quantiles of the pivotal quantities are provided. Numerical examples are also given for illustration purposes.

Keywords and Phrases: Laplace distribution; exact prediction intervals; predictive likelihood function; maximum likelihood estimators; ratios of linear combinations of exponential random variables

1 Introduction

Let $X_1, \ldots, X_n$, $n \geq 2$, be a random sample from the Laplace (double exponential) distribution $\mathcal{L}(\mu, \sigma)$, with probability density function (pdf)

$$f(x; \mu, \sigma) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma}, \quad x \in \mathbb{R}, \tag{1}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ are unknown parameters. Let also $X_{1:n} < \cdots < X_{n:n}$ denote the corresponding order statistics. Suppose that the sample is censored, that is, we have observed only $X_{r+1:n} < \cdots < X_{n-s:n}$ where $r, s \geqslant 0$. When at least one among r and s is positive, Type-II censoring occurs; if $r > 0$, $s = 0$ there is left censoring, if $r = 0$, $s > 0$ there is right censoring, and if both are positive there is double censoring. In order to be

*Corresponding author; Department of Statistics and Insurance Science, University of Piraeus, 80 Karaoli & Dimitriou str., 18534 Piraeus, Greece, e–mail: geh@unipi.gr

†Department of Statistics, Faculty of Basic Science, University of Mazandaran, Post Code 47416-95447, Babolsar, Iran, e–mail: m.mirmostafaee@umz.ac.ir

able to estimate the unknown parameters we need at least two observations, therefore we will assume that $n-r-s \geqslant 2$.

Childs and Balakrishnan (1997) (see also Balakrishnan and Cutler, 1995) showed that the maximum likelihood estimators (MLEs) of the parameters based on such a doubly Type-II censored sample are given in closed form. Let $m=(n+1)/2$ if the sample size is odd and $n/2$ if it is even and denote by $[x]$ the integer part of x. The MLEs of σ and μ are

$$\hat{\sigma}_{\mathrm{MLE}} = \begin{cases} \frac{1}{n-r-s}\Big\{\sum\limits_{i=m+1}^{n-s} X_{i:n} + sX_{n-s:n} - rX_{r+1:n} - \sum\limits_{i=r+1}^{[n/2]} X_{i:n}\Big\}, & \text{if } \max(r,s)<m, \\ \frac{1}{n-r-s}\Big\{\sum\limits_{i=r+1}^{n-s} (X_{i:n}-X_{r+1:n}) + s(X_{n-s:n}-X_{r+1:n})\Big\}, & \text{if } r \geqslant m, \\ \frac{1}{n-r-s}\Big\{\sum\limits_{i=r+1}^{n-s} (X_{n-s:n}-X_{i:n}) + r(X_{n-s:n}-X_{r+1:n})\Big\}, & \text{if } s \geqslant m, \end{cases}$$

and

$$\hat{\mu}_{\mathrm{MLE}} = \begin{cases} \text{any median of } X\text{'s}, & \text{if } \max(r,s)<m, \\ X_{r+1:n} - \hat{\sigma}_{\mathrm{MLE}} \log\left(\frac{n}{2(n-r)}\right), & \text{if } r \geqslant m, \\ X_{n-s:n} + \hat{\sigma}_{\mathrm{MLE}} \log\left(\frac{n}{2(n-s)}\right), & \text{if } s \geqslant m. \end{cases}$$

When $n=2m-1$, the median of X's is uniquely defined; it is $X_{m:n}$. On the other hand, if $n=2m$, any point in the interval $[X_{m:n}, X_{m+1:n}]$ is a median of X's. However, it is standard to define the midpoint of this interval as the sample median so hereafter we set $\hat{\mu}_{\mathrm{MLE}} = (X_{m:n}+X_{m+1:n})/2$ when $n=2m$ and $\max(r,s)<m$. Note also that in all cases it holds $\hat{\sigma}_{\mathrm{MLE}} = \sum_{i=r+1}^{n-s} |X_{i:n} - \hat{\mu}_{\mathrm{MLE}}|/(n-r-s)$.

Many researchers have studied inferential procedures for the Laplace distribution. Bain and Engelhardt (1973) derived approximate confidence intervals for the two parameters based on a complete sample. Kappenman (1975) constructed conditional exact confidence intervals by conditioning on appropriate ancillary statistics. Childs and Balakrishnan (1996, 2000) extended Kappenman's (1975) results in the case of Type-II right censored and progressively Type-II right censored data, respectively. Balakrishnan and Chandramouleeswaran (1996) estimated the reliability function and found tolerance limits based on Type-II right censored samples. For more results and developments on the Laplace distribution in the 20$^{\text{th}}$ century we refer to the book by Kotz, Kozubowski and Podgórski (2001). Recently, Iliopoulos and Balakrishnan (2011) considered Type-II doubly censored samples and derived the exact distributions of the MLEs-based pivotal quantities for μ and σ. They also tabulated the most typical quantiles so that they are readily available for the formulation of exact confidence intervals or performing tests of hypotheses for the two parameters of the distribution.

The purpose of this paper is to derive exact prediction intervals for unobservable (either future or past and missing) order statistics from the Laplace distribution on the basis of a (possibly doubly) Type-II censored sample. We discuss both the one-sample and two-sample cases. In the one-sample case, we assume that we have observed $X_{r+1:n} < \cdots < X_{n-s:n}$ with $\max(r, s) > 0$ and we construct prediction intervals for the unobserved order statistics $X_{n-s+k:n}$, $k \in \{1, \ldots, s\}$ (if $s > 0$), or $X_{r+1-k:n}$, $k \in \{1, \ldots, r\}$ (if $r > 0$). The corresponding prediction intervals are based on the pivotal quantities

$$T_1 \equiv T_1(n, r, s, k) = \frac{X_{n-s+k:n} - X_{n-s:n}}{\hat{\sigma}_{\text{MLE}}} \quad \text{or} \quad T_2 \equiv T_2(n, r, s, k) = \frac{X_{r+1:n} - X_{r+1-k:n}}{\hat{\sigma}_{\text{MLE}}}, \tag{2}$$

respectively. In the two-sample case we make also the same assumption (although here it may hold $r = s = 0$, i.e., the observed data to be complete) and we construct prediction intervals for order statistics from a future sample $Y_1, \ldots, Y_{n'}$ from the same distribution. In this case, the prediction interval for the order statistic $Y_{k:n'}$ is based on the pivotal quantity

$$T_3 \equiv T_3(n, r, s, n', k) = \frac{Y_{k:n'} - \hat{\mu}_{\text{MLE}}}{\hat{\sigma}_{\text{MLE}}}. \tag{3}$$

Clearly, the distributions of the above quantities are parameter-free. They have also an obvious intuitive interpretation. However, in Section 2 we further justify their use in terms of maximum likelihood prediction.

In the past, prediction intervals for certain order statistics from the Laplace distribution based on censored samples had been derived by Balakrishnan and Chandramouleeswaran (1994). These authors used pivotal quantities as in (2), (3) (but with the best linear unbiased estimators of μ and σ in the place of MLEs) and approximated the required quantiles by a limited number of Monte Carlo simulations. The prediction intervals we construct here are exact in the sense that they are based on the exact distributions of the above pivotal quantities. The distributions are derived in Sections 3 and 4. For this purpose we follow the approach of Iliopoulos and Balakrishnan (2011) and express these distributions as mixtures of distributions of ratios of linear combinations of independent standard exponential random variables. Subsequently, in Section 5 practical issues about the calculation of the prediction intervals are discussed while in Section 6 illustrative examples are presented. Finally, an appendix contains results related to the afore-mentioned distributions of ratios which are also of independent interest.

2 One- and two-sample maximum likelihood prediction

Kaminsky and Rhodin (1985) extended the notion of the likelihood function in order to allow for prediction of future (or past yet missing) observations. The *predictive likelihood function* (PLF) is defined to be the joint probability density function of observed and unobserved quantities considered as a function of the latter and the unknown parameters. The maximizer with respect to the quantities we wish to predict is the *maximum likelihood predictor* (MLP) while the maximizer with respect to the parameters is the *predictive maximum likelihood estimator* (PMLE). Note that the PMLE does not necessarily coincide with the usual MLE. It is also worth mentioning the fact that the main purpose of Kaminsky and Rhodin (1985) when suggested the PLF approach was to apply it to the problem of prediction of future order statistics.

For convenience, in this section we denote $x_{i:n}$ and $y_{k:n'}$ simply by x_i and y_k, respectively.

2.1 One-sample prediction

Assume that we have observed $(X_{r+1:n}, \ldots, X_{n-s:n}) = (x_{r+1}, \ldots, x_{n-s})$, where $s > 0$, and we want to predict X_{n-s+k} for some $k = 1, \ldots, s$. Then, the predictive likelihood function is

$$\begin{aligned} L \equiv L(x_{n-s+k}, \mu, \sigma | x_{r+1}, \ldots, x_{n-s}) \propto \\ F(x_{r+1})^r \left\{ \prod_{i=r+1}^{n-s} f(x_i) \right\} \{F(x_{n-s+k}) - F(x_{n-s})\}^{k-1} f(x_{n-s+k}) \{1 - F(x_{n-s+k})\}^{s-k}, \\ x_{n-s+k} \geqslant x_{n-s},\ \mu \in \mathbb{R},\ \sigma > 0. \end{aligned}$$

Since

$$F(x) = \begin{cases} \frac{1}{2}\, e^{(x-\mu)/\sigma}, & x < \mu \\ 1 - \frac{1}{2} e^{(\mu-x)/\sigma}, & x \geqslant \mu \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \frac{1}{2\sigma}\, e^{(x-\mu)/\sigma}, & x < \mu \\ \frac{1}{2\sigma} e^{(\mu-x)/\sigma}, & x \geqslant \mu, \end{cases}$$

the PLF takes the following forms depending on the position of μ relative to the x's:

(a) For $\mu < x_{r+1}$,

$$L = e^{(n-r)\mu/\sigma} \{2 - e^{(\mu - x_{r+1})/\sigma}\}^r e^{-(s-k+1)x_{n-s+k}/\sigma} (e^{-x_{n-s}/\sigma} - e^{-x_{n-s+k}/\sigma})^{k-1} \frac{e^{-\sum_{i=r+1}^{n-s} x_i/\sigma}}{2^n \sigma^{n-r-s+1}};$$

(b) for $x_{r+1} \leqslant \cdots \leqslant x_d < \mu \leqslant x_{d+1} \leqslant \cdots \leqslant x_{n-s}$ for some $d = r+1, \ldots, n-s-1$,

$$L = e^{(n-2d)\mu/\sigma} e^{-(s-k+1)x_{n-s+k}/\sigma} (e^{-x_{n-s}/\sigma} - e^{-x_{n-s+k}/\sigma})^{k-1} \frac{e^{(r x_{r+1} + \sum_{i=r+1}^{d} x_i - \sum_{i=d+1}^{n-s} x_i)/\sigma}}{2^n \sigma^{n-r-s+1}};$$

(c) for $x_{n-s} < \mu \leqslant x_{n-s+k}$,

$$L = e^{-(n-2s+k-1)\mu/\sigma}\{2-e^{(\mu-x_{n-s+k})/\sigma}-e^{(x_{n-s}-\mu)/\sigma}\}^{k-1}e^{-(s-k+1)x_{n-s+k}/\sigma}\frac{e^{(rx_{r+1}+\sum_{i=r+1}^{n-s}x_i)/\sigma}}{2^n\sigma^{n-r-s+1}};$$

(d) for $\mu > x_{n-s+k}$,

$$L = e^{-(n-s+k)\mu/\sigma}\{2-e^{(x_{n-s+k}-\mu)/\sigma}\}^{s-k}e^{x_{n-s+k}/\sigma}(e^{x_{n-s+k}/\sigma}-e^{x_{n-s}/\sigma})^{k-1}\frac{e^{(rx_{r+1}+\sum_{i=r+1}^{n-s}x_i)/\sigma}}{2^n\sigma^{n-r-s+1}}.$$

Notice that for any fixed σ and x_{n-s+k}, L is continuous in μ.

We will first maximize L with respect to μ and x_{n-s+k} and then plug the estimates in and maximize with respect to σ. For convenience we will set $\delta = \mu/\sigma$, $z_i = x_i/\sigma$ for all i. This replacement will not affect the first maximization process. However, we need to remember that before proceeding to the next step, the maximizers with respect to δ and z_{n-s+k} which will be denoted by $\hat{\delta}$ and $\hat{z}_{n-s+k}$, respectively, must be multiplied by σ.

Case $\max(r, s) < m$

Note first that $\max(r, s) < m$ implies that both $n - 2r$ and $n - 2s$ are strictly positive. When $\delta < z_{r+1}$, the derivative of $\log L$ with respect to δ is $n - r - r/(2e^{z_{r+1}-\delta} - 1) > n - 2r$, since $e^{z_{r+1}-\delta} > 1$. Hence, L is strictly increasing in δ for $\delta \in (-\infty, z_{r+1})$. When $\delta \in (z_{n-s}, z_{n-s+k})$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\log L = -(n-2s) - 2(k-1)\frac{1-e^{z_{n-s}-\delta}}{(1-e^{z_{n-s}-\delta}) + (1-e^{\delta-z_{n-s+k}})}. \tag{4}$$

Since both $e^{z_{n-s}-\delta}$ and $e^{\delta-z_{n-s+k}}$ are less than one, the fraction is between 0 and 1 and so, this quantity is $\leqslant -(n-2s)$. Thus, L is strictly decreasing in δ for $\delta \in (z_{n-s}, z_{n-s+k})$. For $\delta \in [z_{n-s+k}, \infty)$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\log L = -(n-s+k) + \frac{s-k}{2e^{\delta-z_{n-s+k}}-1} \leqslant -(n-2s+2k) < -(n-2s) \tag{5}$$

since $e^{\delta-z_{n-s+k}} \geqslant 1$. So, L remains strictly decreasing in this interval as well. Consider finally the case $\delta \in [z_{r+1}, z_{n-s}]$. Here L is strictly increasing for $\delta \leqslant z_{[n/2]}$ and strictly decreasing for $\delta \geqslant z_{[n/2]+1}$. What happens in between it depends on whether n is odd or even: If $n = 2m - 1$ then $[n/2] = m - 1$ and so the maximum is attained at z_m while if $n = 2m$ L remains constant in this interval and thus any point inthere corresponds to a maximum. Hence, in this case $\hat{\delta}$ is any median of z's. When we replace δ by the median we fall in the case (b) of the previous list. It is easy to verify that the maximum with respect to z_{n-s+k} occurs at $z_{n-s} + \log\{s/(s-k+1)\}$.

Case $s \geqslant m$

Again, for $\delta < z_{r+1}$ L is strictly increasing. The same happens for $\delta \in [z_{r+1}, z_{n-s}]$ as well since $n > 2d$ for all $d \leqslant n-s-1$. Hence, the maximizer with respect to δ must be larger than or equal to z_{n-s}. Now we consider the following cases:

Case $n-2s=0,\ k=1$: In this case L is constant with respect to δ when $\delta \in (z_{n-s}, z_{n-s+1}]$, while for $\delta > z_{n-s+k}$ we have $\frac{\mathrm{d}}{\mathrm{d}\delta}\log L = -(s+1) + (s-1)/(2e^{\delta - z_{n-s+1}} - 1) < -2$, since $e^{\delta - z_{n-s+1}} > 1$. Hence, L achieves its maximum with respect to δ at any point between z_{n-s} and z_{n-s+1}. On the other hand, the derivative of $\log L$ with respect to z_{n-s+1} for $\delta \in [z_{n-s}, z_{n-s+1}]$ equals $-s < 0$ and this means that it is maximized for $z_{n-s+1} = z_{n-s}$. Hence, we have $\hat{\delta} = \hat{z}_{n-s+k} = z_{n-s}$.

Case $n-2s=0,\ k>1$: Consider first $\delta \in (z_{n-s}, z_{n-s+k})$. Then, the derivative in (4) becomes $2(k-1)(e^{z_{n-s}-\delta} - 1)/(2 - e^{\delta - z_{n-s+k}} - e^{z_{n-s}-\delta}) < 0$ so $\log L$ is strictly decreasing for such δ. Next, for $\delta \geqslant z_{n-s+k}$ we have $\frac{\mathrm{d}}{\mathrm{d}\delta}\log L = -(s+k) + (s-k)/(2e^{\delta - z_{n-s+k}} - 1) < -2k < 0$ so it is strictly decreasing as well. This implies that the maximum with respect to δ occurs at z_{n-s}. By replacing $\delta = z_{n-s}$ we see that $\frac{\mathrm{d}}{\mathrm{d}z_{n-s+k}}\log L = (k-1)/(e^{z_{n-s+k}-z_{n-s}} - 1) - (s-k+1)$ and by equating it to zero we get $\hat{z}_{n-s+k} = z_{n-s} + \log\{s/(s-k+1)\}$.

Notice that the previous cases occur when $n = 2m$ and $s = m$.

Case $n-2s<0$: Note first that the limit of the derivative in (4) as $\delta \downarrow z_{n-s}$ equals $-(n-2s)$. This means that the maximizer must satisfy $\hat{\delta} > z_{n-s}$. Observe also that the derivative is always $> -\{n-2s+2(k-1)\}$. This follows from the fact that the fraction is between 0 and 1. Hence, if $n-2s+2(k-1) \leqslant 0$, the maximizer must further satisfy $\hat{\delta} \geqslant z_{n-s+k}$. Moreover, the derivative in (5) is positive, zero, or negative according to whether δ is smaller, equal, or larger than $z_{n-s+k} + \log\{n/[2(n-s+k)]\}$, respectively. The latter is larger than (resp., equal to) z_{n-s+k} if and only if $n-2s+2k$ is negative (resp., zero). Hence we need to distinguish between the following three cases:

(a) When $n-2s+2(k-1) > 0$ (which is equivalent to $s < m-1+k$ and can hold only if $k > 1$), L is first increasing and then decreasing with respect to δ in (z_{n-s}, z_{n-s+k}); it also keeps decreasing in $[z_{n-s+k}, \infty)$. Hence the maximizer satisfies $z_{n-s} < \hat{\delta} < \hat{z}_{n-s+k}$.

(b) When $n-2s+2(k-1) \leqslant 0 \leqslant n-2s+2k$ (which is equivalent to $m-1+k \leqslant s \leqslant [n/2]+k$), L is increasing with respect to δ in (z_{n-s}, z_{n-s+k}) and decreasing in $[z_{n-s+k}, \infty)$. Hence, the maximizer satisfies $z_{n-s} < \hat{\delta} = \hat{z}_{n-s+k}$.

(c) When $n-2s+2k<0$ (which is equivalent to $s \geqslant [n/2]+1+k$), L is increasing with respect to δ in (z_{n-s}, z_{n-s+k}), keeps increasing as δ enters the interval $[z_{n-s+k}, \infty)$ until some point after which it decreases. Hence, the maximizer satisfies $z_{n-s} < \hat{z}_{n-s+k} < \hat{\delta}$.

After having observed the above facts, the maximization becomes a simple procedure: Under cases (a) and (c) we equate the derivatives with respect to δ and z_{n-s+k} simultaneously to zero and solve the system while under case (b) we set $\delta = z_{n-s+k}$ and we maximize with just one "parameter". We do not describe the details of the derivations here; we just report the maximizers. In case (a) we have $\hat{\delta} = z_{n-s} + \log\{n/[2(n-s)]\}$, $\hat{z}_{n-s+k} = z_{n-s} + \log\{n^2/[4(n-s)(s-k+1)]\}$, in case (b) $\hat{\delta} = \hat{z}_{n-s+k} = z_{n-s} + \log\{(n-s+k-1)/(n-s)\}$, and in case (c) $\hat{\delta} = z_{n-s} + \log\{n(n-s+k-1)/[2(n-s)(n-s+k)]\}$, $\hat{z}_{n-s+k} = z_{n-s} + \log\{(n-s+k-1)/(n-s)\}$.

Case $r \geqslant m$

Note that $r \geqslant m$ implies $r \geqslant n/2$. When $\delta < z_{r+1}$ the derivative of $\log L$ with respect to δ is first positive and then negative, and it becomes zero at $\delta = z_{r+1} - \log\{n/[2(n-r)]\}$. For $\delta \geqslant z_{r+1}$ we always have $d \geqslant r+1 > n/2$ and so, L is strictly decreasing. It can be verified that the same is true for $\delta \geqslant z_{n-s}$ as well. Hence, $\hat{\delta} = z_{r+1} - \log\{n/[2(n-r)]\}$ and since this maximizer is smaller than z_{n-s} we get $\hat{z}_{n-s+1} = z_{n-s} + \log\{s/(s-k+1)\}$.

Finally, we maximize L with respect to σ. First, we replace μ, x_{n-s+k} by the corresponding maximizers $\hat{\mu} = \sigma\hat{\delta}$, $\hat{x}_{n-s+k} = \sigma\hat{z}_{n-s+k} = x_{n-s} + c(n,s,k)\sigma$, where the constant $c(n,s,k)$ depends on the particular configuration of (n,s,k). It is easy to verify that the (profile) likelihood becomes proportional to $\sigma^{-(n-r-s+1)} \exp\{-\sum_{i=r+1}^{n-s} |x_i - \hat{\mu}|/\sigma\}$ and hence, it is maximized at $\hat{\sigma} = \sum_{i=r+1}^{n-s} |x_i - \hat{\mu}|/(n-r-s+1)$.

The next proposition summarizes the results of this section.

Proposition 1. *The predictive MLEs of μ and σ are*

$$\hat{\mu} = \begin{cases} \textit{any sample median}, & \max(r,s) < m, \\ X_{r+1:n} - \hat{\sigma}\log\left(\frac{n}{2(n-r)}\right), & r \geqslant m, \\ X_{n-s:n} + \hat{\sigma}\log\left(\frac{n}{2(n-s)}\right), & m \leqslant s < m-1+k \textit{ and } k>1, \\ X_{n-s:n} + \hat{\sigma}\log\left(\frac{n-s+k-1}{n-s}\right), & m-1+k \leqslant s \leqslant [n/2]+k, \\ X_{n-s:n} + \hat{\sigma}\log\left(\frac{n(n-s+k-1)}{2(n-s)(n-s+k)}\right), & s \geqslant [n/2]+1+k \end{cases}$$

and $\hat{\sigma} = \sum\limits_{i=r+1}^{n-s} |X_{i:n} - \hat{\mu}|/(n-r-s+1)$, *respectively. The MLP of* $X_{n-s+k:n}$ *is* $\hat{X}_{n-s+k:n} =$

$X_{n-s:n} + c(n,s,k)\hat{\sigma}$, *where*

$$c(n,s,k) = \begin{cases} \log\left(\frac{s}{s-k+1}\right), & s < m, \\ \log\left(\frac{n^2}{4(n-s)(s-k+1)}\right), & m \leqslant s < m-1+k \text{ and } k > 1, \\ \log\left(\frac{n-s+k-1}{n-s}\right), & s \geqslant m-1+k. \end{cases}$$

The natural pivotal quantity that is suggested by the predictive likelihood approach is $T = (X_{n-s+k:n} - \hat{X}_{n-s+k:n})/\hat{\sigma}$. Note, however, that $T = c(n,s,k) + (X_{n-s+k:n} - X_{n-s:n})/\hat{\sigma}$ which means that we can rely just on the second term. Moreover, $\hat{\sigma}$ is proportional to $\hat{\sigma}_{\text{MLE}}$ because it always holds $\sum |X_{i:n} - \hat{\mu}| = \sum |X_{i:n} - \hat{\mu}_{\text{MLE}}|$. Hence, the prediction intervals that arise from T are exactly the same as those we will get if we use $T_1(n,r,s,k)$ in (2).

Assume now that $r > 0$ and that we want to predict $X_{r+1-k:n}$ for some $k = 1, \dots, r$. Due to the symmetry of the Laplace distribution, the solution to this problem is essentially the same as before with the roles of r and s reversed. To be more specific, consider the transformation $X'_i = -X_i$, $i = 1, \dots, n$, which implies that $X'_1, \dots, X'_n$ is a random sample from $\mathcal{L}(-\mu, \sigma)$. Now our actual observations can be expressed as $X'_{s+1:n} < \cdots < X'_{n-r:n}$ and the order statistic to be predicted is $X'_{n-r+k:n}$. Arguing as before, we get the pivotal quantity $(X'_{n-r+k:n} - X'_{n-r})/\hat{\sigma}_{\text{MLE}}$. Then, by switching back to the X's we see that this is the same as $T_2(n,r,s,k)$ in (2).

2.2 Two-sample prediction

For the prediction of $Y_{k:n'}$ the PLF becomes

$$L \equiv L(y, \mu, \sigma | x_{r+1}, \dots, x_{n-s}) \propto$$
$$F(x_{r+1})^r \left\{ \prod_{i=r+1}^{n-s} f(x_i) \right\} \{1 - F(x_{n-s})\}^s f(y) F(y)^{k-1} \{1 - F(y)\}^{n'-k}, \quad y, \mu \in \mathbb{R}, \sigma > 0.$$

As we will see below this is a much easier task than that in the previous section.

The following lemma is helpful for the maximization of L. Its proof is elementary and so, it is omitted.

Lemma 1. *Let* $Y_1, \dots, Y_{n'} \overset{\text{iid}}{\sim} \mathcal{L}(\mu, \sigma)$. *The pdf of* $Y_{k:n'}$ *has mode* $\mu + c'(n', k)\sigma$, *where*

$$c'(n',k) = \begin{cases} -\log\left(\frac{n'}{2k}\right), & \text{if } k < n'/2, \\ 0, & \text{if } n'/2 \leqslant k \leqslant n'/2+1, \\ \log\left(\frac{n'}{2(n'-k+1)}\right), & \text{if } k > n'/2+1. \end{cases} \tag{6}$$

In our case Lemma 1 implies that once we get $\hat{\mu}$ and $\hat{\sigma}$ we immediately have the maximizer with respect to y as a linear function of them. Now, it can be readily verified that if we set $y = \mu + c\sigma$ in L, then its part containing y becomes proportional to $1/\sigma$. Hence, at the next step we have to maximize the same function as in Childs and Balakrishnan (1997) with the only difference that in the denominator, σ is raised to the $(n-r-s+1)$th power instead of the $(n-r-s)$th. So, in this case the maximizer with respect to μ coincides with $\hat{\mu}_{\mathrm{MLE}}$ while the maximizer with respect to σ is a scaled version of $\hat{\sigma}_{\mathrm{MLE}}$. Summarizing, we get the following result.

Proposition 2. *The predictive MLEs of μ and σ are $\hat{\mu} = \hat{\mu}_{\mathrm{MLE}}$ and $\hat{\sigma} = \frac{n-r-s}{n-r-s-1}\,\hat{\sigma}_{\mathrm{MLE}}$. The MLP of $Y_{k:n'}$ is $\hat{Y}_{k:n'} = \hat{\mu} + c'(n',k)\hat{\sigma}$ where $c'(n',k)$ is defined in* (6)*.*

It turns out that $T_3(n,r,s,n',k)$ in (3) is the pivotal quantity suggested by the predictive likelihood approach up to a multiplicative constant.

3 The exact distributions of $T_1(n,r,s,k)$ and $T_2(n,r,s,k)$

3.1 Some preliminary facts

Let $X_1,\ldots,X_n$ be a random sample from any parent distribution and let D denote the number of X's that are at most equal to a pre-fixed constant C. Iliopoulos and Balakrishnan (2009) proved that conditional on $D = d$, where $d \in \{0,1,\ldots,n\}$, the blocks of order statistics $(X_{1:n},\ldots,X_{d:n})$ and $(X_{d+1:n},\ldots,X_{n:n})$ are independent. Moreover, conditional on $D = d$, $(X_{1:n},\ldots,X_{d:n}) \stackrel{\mathrm{d}}{=} (L_{d:d},\ldots,L_{1:d})$ and $(X_{d+1:n},\ldots,X_{n:n}) \stackrel{\mathrm{d}}{=} (R_{1:n-d},\ldots,R_{n-d:n-d})$, where "$\stackrel{\mathrm{d}}{=}$" denotes equality in distribution, $L_1,\ldots,L_d$ is a random sample from the parent distribution right truncated at C, and $R_1,\ldots,R_{n-d}$ is a random sample from the parent distribution left truncated at C.

It is well-known that when $X \sim \mathcal{L}(\mu,\sigma)$, the conditional distribution of $\mu - X$ given $X \leqslant \mu$ and the conditional distribution of $X-\mu$ given $X > \mu$ (i.e., the Laplace distribution right and left truncated at its median) are both exponentials with scale σ. We will denote this distribution by $\mathcal{E}(\sigma)$. Let now $X_1,\ldots,X_n \stackrel{\mathrm{iid}}{\sim} \mathcal{L}(\mu,\sigma)$ and $D = \#\{X's \leqslant \mu\}$. In light of the above result we conclude that, conditional on $D = d$ ($d \in \{0,1,\ldots,n\}$), the blocks $(X_{1:n},\ldots,X_{d:n})$, $(X_{d+1:n},\ldots,X_{n:n})$ are independent and $(\mu - X_{1:n},\ldots,\mu - X_{d:n}) \stackrel{\mathrm{d}}{=} (L_{d:d},\ldots,L_{1:d})$, $(X_{d+1:n} - \mu,\ldots,X_{n:n} - \mu) \stackrel{\mathrm{d}}{=} (R_{1:n-d},\ldots,R_{n-d:n-d})$, where $L_1,\ldots,L_d$ and $R_1,\ldots,R_{n-d}$ are independent $\mathcal{E}(\sigma)$ random variables.

3.2 The exact distribution of $T_1(n,r,s,k)$

In this section we will derive the exact distribution of $T_1(n,r,s,k)$ in (2). Since its distribution is free of the parameters we may (and will) take without loss of generality $\mu = 0$, $\sigma = 1$.

Let $D = \#\{X's \leqslant 0\}$. Clearly, the random variable D follows the binomial distribution $\mathcal{B}(n, 1/2)$. We will derive the conditional distribution of T_1 given $D = d$ for all $d \in \{0, 1, \ldots, n\}$ and then express its (unconditional) distribution as a mixture over D. More specifically, if we set $P(T_1 \leqslant t|D = d) \equiv F^{(1)}(t|d)$, then the cumulative distribution function (cdf) of T_1 is

$$P(T_1 \leqslant t) \equiv F^{(1)}(t) = \frac{1}{2^n}\sum_{d=0}^{n}\binom{n}{d}F^{(1)}(t|d), \quad t \geqslant 0. \tag{7}$$

Iliopoulos and Balakrishnan (2011) used the same approach in order to derive the distribution of MLE-based pivotal quantities for the parameters μ and σ.

Note that for notation simplicity it is more convenient to derive the conditional distribution of $T_1/(n-r-s)$ given $D = d$. This takes different forms according to whether $\max(r,s) < m$, $s \geqslant m$ of $r \geqslant m$. Table 1 contains the complete list for all d. Due to space limitations we will give here the details of the derivation for one particular case; all other cases' derivations can be found in a supplementary file at `http://www.unipi.gr/faculty/geh/laplace.prediction/supplementary.htm`.

Table 1 comes here

In what follows $L_1, L_2, \ldots$ and $R_1, R_2, \ldots$ denote independent exponential random variables. Moreover, the corresponding spacings of order statistics are denoted by $\tilde{L}$ and $\tilde{R}$, respectively: For any k we set $\tilde{L}_{i:k} = L_{i:k} - L_{i-1:k}$ and $\tilde{R}_{i:k} = R_{i:k} - R_{i-1:k}$. When $i = 1$ we simply set $\tilde{L}_{1:k} = L_{1:k}$ and $\tilde{R}_{1:k} = R_{1:k}$. Recall finally the following property of order statistics from the exponential distribution: The normalized spacings $k\tilde{R}_{1:k}, (k-1)\tilde{R}_{2:k}, \ldots, (k-i+1)\tilde{R}_{i:k}, \ldots, R_{k:k}$ are mutually independent and have the same exponential distribution with R's (see Arnold, Balakrishnan and Nagaraja, 2008).

Consider the case $\max(r,s) < m$ and let $d \in (r, m-1]$. Then, conditional on $D = d$ we have

$$\begin{aligned} T_1/(n-r-s) &= \frac{X_{n-s+k:n} - X_{n-s:n}}{\hat{\sigma}_{\text{MLE}}} \\ &= \frac{X_{n-s+k:n} - X_{n-s:n}}{-rX_{r+1:n} - \sum_{i=r+1}^{d} X_{i:n} - \sum_{i=d+1}^{[n/2]} X_{i:n} + \sum_{m+1}^{n-s} X_{i:n} + sX_{n-s:n}} \end{aligned}$$

(by making the convention $\sum_{i=k}^{\ell} \equiv 0$ when $k > \ell$)

$$\stackrel{\mathrm{d}}{=} \frac{R_{n-s+k-d:n-d} - R_{n-s-d:n-d}}{rL_{d-r:d} + \sum_{i=1}^{d-r} L_{i:n} - \sum_{i=1}^{[n/2]-d} R_{i:n-d} + \sum_{m+1-d}^{n-s-d} R_{i:n-d} + sR_{n-s-d:n-d}}$$

$$= \frac{\sum_{i=n-s-d+1}^{n-s-d+k} \tilde{R}_{i:n-d}}{\sum_{i=1}^{d-r}(d-i+1)\tilde{L}_{i:d} + \sum_{i=1}^{m-d}(d+i-1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-s-d}(n-d-i+1)\tilde{R}_{i:n-d}}.$$

$$\stackrel{\mathrm{d}}{=} \frac{\sum_{i=n-s-d+1}^{n-s-d+k} \frac{1}{n-d-i+1} R_i}{\sum_{i=1}^{d-r} L_i + \sum_{i=1}^{m-d} \frac{d+i-1}{n-d-i+1} R_i + \sum_{i=m-d+1}^{n-s-d} R_i},$$

It can be verified that the last fraction has cdf $F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a)$, see Proposition 3(a) in the Appendix, with $p = k$, $q = m - d$, $\boldsymbol{\eta} = (s-k+1, \ldots, s)$, $\boldsymbol{\theta} = ((n-d)/d, \ldots, (n-m+1)/(m-1))$, and $a = n - m - s - r + d$.

3.3 The exact distribution of $T_2(n, r, s, k)$

Suppose that $r \geqslant 1$. Then it may be of interest to predict $X_{r+1-k:n}$, $k = 1, \ldots, r$. The distribution of the corresponding pivot T_2 which is introduced in (2), may be derived as that of T_1. However, the symmetry of the standard Laplace distribution about zero implies that $(X_{1:n}, \ldots, X_{n:n}) \stackrel{\mathrm{d}}{=} (-X_{n:n}, \ldots, -X_{1:n})$ and thus, $T_2(n, r, s, k) \stackrel{\mathrm{d}}{=} T_1(n, s, r, k)$. Hence, the cdf $F^{(2)}$ of T_2 is as in (7) but with r, s interchanged.

4 The exact distribution of $T_3(n, r, s, n', k)$

Let us now proceed to determine the exact distribution of $T_3(n, r, s, n', k)$ in (3). This distribution is also free of the parameters μ and σ and thus we may again take without loss of generality $\mu = 0$ and $\sigma = 1$. Moreover, in addition to the random variable D defined in the previous section, we introduce the corresponding random variable related to the future sample. So, let $D' = \#\{Y's \leqslant 0\}$. Obviously, D' can be taken independent of D. Below we will determine the conditional distribution of T_3 given $D = d$ *and* $D' = d'$ for all pairs $(d, d') \in \{0, 1, \ldots, n\} \times \{0, 1, \ldots, n'\}$. If we set $P(T_3 \leqslant t | D = d, D' = d') = F^{(3)}(t|d, d')$, then we have

$$P(T_3 \leqslant t) \equiv F^{(3)}(t) = \frac{1}{2^{n+n'}} \sum_{d=0}^{n} \sum_{d'=0}^{n'} \binom{n}{d} \binom{n'}{d'} F^{(3)}(t|d, d'), \quad t \in \mathbb{R}. \tag{8}$$

In order to derive the conditional distributions of T_3 given $D = d$, $D' = d'$ we introduce two additional independent sequences of iid standard exponential random variables which will be denoted by $L_1', L_2', \ldots$ and $R_1', R_2', \ldots$. These sequences will be related with the second sample and so they will be considered independent of the L- and R-sequences introduced in the previous section. The corresponding spacings will be denoted by $\tilde{L}'$ and $\tilde{R}'$, respectively.

For further use, observe that conditional on $D' = d'$ (and $D = d$),

$$Y_{k:n'} \overset{\mathrm{d}}{=} \begin{cases} R'_{k-d':n'-d'}, & 0 \leqslant d' < k \\ -L'_{d'-k+1:d'}, & k \leqslant d' \leqslant n' \end{cases} = \begin{cases} \sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'}, & 0 \leqslant d' < k \\ -\sum_{i=1}^{d'-k+1} \tilde{L}'_{i:d'}, & k \leqslant d' \leqslant n' \end{cases}$$

$$\overset{\mathrm{d}}{=} \begin{cases} \sum_{i=1}^{k-d'} (n' - d' - i + 1)^{-1} R_i, & 0 \leqslant d' < k \\ -\sum_{i=1}^{d'-k+1} (d' - i + 1)^{-1} L_i, & k \leqslant d' \leqslant n'. \end{cases}$$

Case $\max(r, s) < m$

Due to space limitations we also provide the details for the derivation of the conditional distribution of $T_3/(n-r-s)$ given $D = d$, $D' = d'$ for one particular case. The distributions for the remaining cases are presented in Table 2.

Let $0 \leqslant d \leqslant r$ and $0 \leqslant d' < k$. When $n = 2m - 1$, conditional on $D = d$, $D' = d'$ we may write

$$T_3/(n - r - s) = \frac{Y_{k:n'} - X_{m+1:n}}{-rX_{r+1:n} - \sum_{i=r+1}^{d} X_{i:n} - \sum_{i=d+1}^{[n/2]} X_{i:n} + \sum_{m+1}^{n-s} X_{i:n} + sX_{n-s:n}}$$

$$\overset{\mathrm{d}}{=} \frac{R'_{k-d':n'-d'} - R_{m-d:n-d}}{-rR_{r-d+1:n-d} - \sum_{i=r-d+1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} R_{i:n-d} + sR_{n-d-s:n-d}}$$

$$= \frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d}}{\sum_{i=r-d+2}^{m-d} (d + i - 1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} (n - d - i + 1)\tilde{R}_{i:n-d}}.$$

By normalizing these spacings accordingly, we find out that the cdf of the above fraction is $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$, see Proposition 4(a), with $p = k - d'$, $q = r - d + 1$, $h = m - r - 1$, $\boldsymbol{\eta} = (n' - k + 1, \ldots, n' - d')$, $\boldsymbol{\theta} = (n - r, \ldots, n - d)$, $\boldsymbol{\lambda} = (1/m, \ldots, 1/(n - r - 1))$, $\boldsymbol{\mu} = ((m - 1)/m, \ldots, (r + 1)/(n - r - 1))$, and $a = m - s - 1$. On the other hand, when $n = 2m$ we have, conditional on $D = d$, $D' = d'$, $T_3/(n - r - s)$ to have the same distribution as

$$\frac{R'_{k-d':n'-d'} - \frac{1}{2}R_{m-d:n-d} - \frac{1}{2}R_{m-d+1:n-d}}{-rR_{r-d+1:n-d} - \sum_{i=r-d+1}^{[n/2]-d} R_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} R_{i:n-d} + sR_{n-d-s:n-d}}$$

$$= \frac{\sum_{i=1}^{k-d'} \tilde{R}'_{i:n'-d'} - \sum_{i=1}^{m-d} \tilde{R}_{i:n-d} - \frac{1}{2}\tilde{R}_{m-d+1:n-d}}{\sum_{i=r-d+2}^{m-d} (d + i - 1)\tilde{R}_{i:n-d} + \sum_{i=m-d+1}^{n-d-s} (n - d - i + 1)\tilde{R}_{i:n-d}}.$$

This has also cdf $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$ with p, q, a, $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ as before and $h = m - r$, $\boldsymbol{\lambda} = (1/2m, 1/(m+1), \ldots, 1/(n-r-1))$ and $\boldsymbol{\mu} = (1, (m-1)/(m+1), \ldots, (r+1)/(n-r-1))$.

Table 2 comes here

Cases $s \geqslant m$ and $r \geqslant m$

Note that when $s \geqslant m$,

$$T_3(n,r,s,n',k) = \frac{Y_{k:n'} - \hat{\mu}_{\mathrm{MLE}}}{\hat{\sigma}_{\mathrm{MLE}}} = \frac{Y_{k:n'} - X_{n-s:n}}{\hat{\sigma}} - \log\left(\frac{n}{2(n-s)}\right) \tag{9}$$

and thus, we actually need to find the distribution of $T_3^* = (Y_{k:n'} - X_{n-s:n})/\hat{\sigma}_{\mathrm{MLE}}$. All conditional distributions of $T_3^*/(n-r-s)$ given $D=d$, $D'=d'$ are presented in Table 3.

When $r \geqslant m$, by the symmetry of the standard Laplace distribution about zero we have that $(X_{1:n},\dots,X_{n:n}) \stackrel{\mathrm{d}}{=} (-X_{n:n},\dots,-X_{1:n})$ and $Y_{k:n'} \stackrel{\mathrm{d}}{=} -Y_{n'-k+1:n'}$. It follows that

$$\begin{aligned}
T_3(n,r,s,n',k) &= \frac{Y_{k:n'} - \hat{\mu}_{\mathrm{MLE}}}{\hat{\sigma}_{\mathrm{MLE}}} \\
&= \frac{Y_{k:n'} - X_{r+1:n}}{\left\{\sum_{i=r+1}^{n-s}(X_{i:n} - X_{r+1:n}) + s(X_{n-s:n} - X_{r+1:n})\right\}/(n-r-s)} + \log\left(\frac{n}{2(n-r)}\right) \\
&\stackrel{\mathrm{d}}{=} \frac{-Y_{n'-k+1:n'} + X_{n-r:n}}{\left\{\sum_{i=r+1}^{n-s}(X_{n-r:n} - X_{i:n}) + s(X_{n-r:n} - X_{s+1:n})\right\}/(n-r-s)} + \log\left(\frac{n}{2(n-r)}\right) \\
&\stackrel{\mathrm{d}}{=} -T_3(n,s,r,n',n'-k+1)
\end{aligned}$$

by (9). Thus, the distribution of $-T_3(n,r,s,n',k)$ is obtained from the previous case after switching r with s and replacing k by $n'-k+1$.

Table 3 comes here

5 Construction of exact prediction intervals

In the previous sections we derived the distributions of T_1, T_2 and T_3. In order to construct corresponding prediction intervals for the quantities of interest we need their quantiles. The upper α-quantile of T_j is the solution of the nonlinear equation

$$F^{(j)}(t) = 1 - \alpha \tag{10}$$

with respect to t. Denote by $t_{j,\alpha}(n,r,s,\cdots)$ the upper α-quantile of $T_j(n,r,s,\cdots)$. Then, the typical $100(1-\alpha)\%$ prediction intervals for $X_{n-s+k:n}$, $X_{r+1-k:n}$ and $Y_{k:n'}$ will be

$$\left[X_{n-s:n} + t_{1,1-\alpha/2}(n,r,s,k)\hat{\sigma}_{\mathrm{MLE}},\ X_{n-s:n} + t_{1,\alpha/2}(n,r,s,k)\hat{\sigma}_{\mathrm{MLE}}\right], \tag{11}$$

$$\left[X_{r+1:n} - t_{2,\alpha/2}(n,r,s,k)\hat{\sigma}_{\mathrm{MLE}},\ X_{r+1:n} - t_{2,1-\alpha/2}(n,r,s,k)\hat{\sigma}_{\mathrm{MLE}}\right], \tag{12}$$

and

$$\left[\hat{\mu}_{\mathrm{MLE}} + t_{3,1-\alpha/2}(n,r,s,n',k)\hat{\sigma}_{\mathrm{MLE}},\ \hat{\mu}_{\mathrm{MLE}} + t_{3,\alpha/2}(n,r,s,n',k)\hat{\sigma}_{\mathrm{MLE}}\right], \tag{13}$$

respectively. In a similar manner we can also determine $100(1-\alpha)\%$ prediction bounds for the order statistics of interest. For instance, a $100(1-\alpha)\%$ upper prediction bound for $X_{n-s+k:n}$ is given by $X_{n-s:n}+t_{1,\alpha}(n,r,s,k)\hat{\sigma}_{\mathrm{MLE}}$ while a $100(1-\alpha)\%$ lower prediction bound for $Y_{k:n'}$ is $\hat{\mu}_{\mathrm{MLE}}+t_{3,1-\alpha}(n,r,s,n',k)\hat{\sigma}_{\mathrm{MLE}}$.

Equation (10) can be solved only numerically. Since the complexity of the cdfs increases rapidly with n, at first sight this seems to cause potential problems due to loss of precision. Fortunately, this can be avoided when using the correct software efficiently. Observe that all of the parameters of the conditional cdfs are integers. So, all cdfs $F^{(j)}$ are rational functions, the same being true for their derivatives, i.e., the corresponding pdfs. This implies that if $t \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, then $F^{(j)}(t) \in \mathbb{Q}$ as well. It turns out that packages like `Mathematica` or `Maple` that do not necessarily convert rationals to floating point numbers can evaluate exactly $F^{(j)}(t)$ as long as $t \in \mathbb{Q}$. More importantly, the solution of (10) can be approximated to any desired accuracy if the numerical method that we will use produces a rational sequence.

We considered many configurations of (n,r,s,k) and (n,r,s,n',k) and typical values of α and solved (10) in `Mathematica` using bisection method. In all cases we took the initial interval containing the solution to have rational endpoints and the iterations were continued until its length to become less than 10^{-8}. Then the interval's midpoint was delivered as the corresponding quantile approximation. This guarantees that the reported values are accurate up to (at least) eight decimal places.

Table 4 comes here

Table 4 contains only a few selected quantiles of T_1 just for illustration. The interesting reader can download full tables of quantiles of all pivotal quantities discussed in the paper from `http://www.unipi.gr/faculty/geh/laplace.prediction/predqua.htm`. Moreover, after a referee's suggestion we also conducted a Monte Carlo study to verify that the derived quantiles are exact. The results are shown in Table 4 as well. Indeed, the cdf of T_1 at the reported quantiles has the correct value within the Monte Carlo error limits.

6 Examples

6.1 A simulated dataset

Table 5 contains a simulated dataset of size $n=15$ from $\mathcal{L}(\mu=20,\sigma=5)$. Let us assume that we have at hand only the first ten observations. Then the available data

$x_{1:15}, \ldots, x_{10:15}$ can be thought of as a Type-II right censored sample from a Laplace distribution. Note that, based on these data, the MLE of σ takes the value $\hat{\sigma}_{\text{MLE}} = 4.237$. Suppose now that we want to construct a 95% prediction interval for the "next" observation, $X_{11:15}$ as well as a 95% upper prediction bound for the maximum observations $X_{15:15}$. The upper .025- and .975-quantiles for the configuration $(n, r, s, k) = (15, 0, 5, 1)$ are 0.005561 and 0.9757, respectively; see Table 4. Hence, the 95% prediction interval for $X_{11:15}$ is

$$\begin{aligned} & [x_{10:15} + t_{1,0.975}(15, 0, 5, 1)\hat{\sigma}_{\text{MLE}},\ x_{10:15} + t_{1,0.025}(15, 0, 5, 1)\hat{\sigma}_{\text{MLE}}] \\ = \ & [23.03 + 0.005561 \times 4.237,\ 23.03 + 0.9757 \times 4.237] = [23.054,\ 27.164]; \end{aligned}$$

see (11). On the other hand, the upper .05-quantile for $(n, r, s, k) = (15, 0, 5, 5)$ is 6.1888 and so, the 95% upper prediction bound for $X_{15:15}$ equals

$$x_{10:15} + t_{1,0.05}(15, 0, 5, 5)\hat{\sigma}_{\text{MLE}} = 23.03 + 6.1888 \times 4.237 = 49.252.$$

Table 5 comes here

Suppose now that we have observed only $x_{3:15}, x_{4:15}, x_{5:15}$ and we want to predict $X_{1:15}$. In this case we have $\hat{\sigma}_{\text{MLE}} = 2.160$. The typical form of prediction intervals for "small" order statistics is given in (12). Using also the fact that $t_{2,\alpha}(n, r, s, k) = t_{1,\alpha}(n, s, r, k)$, see Section 3.3, we get the 95% prediction interval for $X_{1:15}$ as

$$\begin{aligned} & [x_{3:5} - t_{2,0.025}(15, 2, 10, 2)\hat{\sigma}_{\text{MLE}},\ x_{3:15} - t_{2,0.025}(15, 2, 10, 2)\hat{\sigma}_{\text{MLE}}] \\ = \ & [17.64 - 21.718 \times 2.160,\ 17.64 - 0.2221 \times 2.160] = [-29.271,\ 17.160]. \end{aligned}$$

6.2 A real data example

Consider the dataset of 33 years of flood data from two stations on For River in Wisconsing, used also by Bain and Engelhardt (1973), Kappenman (1975) and Iliopoulos and Balakrishnan (2011). The data are presented in all of these papers, therefore we do not reproduced them here. By assuming that they come from a Laplace $\mathcal{L}(\mu, \sigma)$ distribution, we estimate the parameters by $\hat{\mu}_{\text{MLE}} = 10.13$, $\hat{\sigma}_{\text{MLE}} = 3.36091$. Suppose now that we want to predict order statistics from a future sample of size $n' = 20$ from the same distribution. Table 6 contains the upper .975- and .025-quantiles of $T_3(33, 0, 0, 20, k)$ for $k = 1, \ldots, 10$. Note that these are sufficient for determining the quantiles for $k = 11, \ldots, 20$ as well since for all s, k it holds $t_{3,\alpha}(n, s, s, n', k) = -t_{3,1-\alpha}(n, s, s, n', n' - k + 1)$. This can be easily proven by using a symmetry argument that applies when the information sample is either complete or symmetrically Type-II censored (i.e., whenever $r = s$). The 95% prediction intervals for $Y_{k:20}$, $k = 1, \ldots, 20$, have been calculated as in (13) and are also displayed in Table 6.

Table 6 comes here

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Appendix

Proposition 3. *Let $U_1, \cdots, U_p, V_1, \cdots, V_q \overset{\text{iid}}{\sim} \mathcal{E}(1)$, $W \sim \mathcal{G}(a, 1)$ be independent random variables (with $a > 0$), $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_p)$, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_q)$ be vectors of positive numbers with the η_i's being distinct. Then we have the following:*

(a) The cdf of $Y = \dfrac{\sum_{j=1}^{p} U_j/\eta_j}{\sum_{j=1}^{q} V_j/\theta_j + W}$ is

$$F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a) = 1 - \sum_{j=1}^{p} \left(\prod_{\substack{i=1 \\ i \neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j} \right) \left(\prod_{i=1}^{q} \frac{\theta_i}{\theta_i + y\eta_j} \right) \frac{1}{(1 + y\eta_j)^a}, \quad y > 0.$$

(b) The cdf of $\dfrac{\sum_{j=1}^{p} U_j/\eta_j}{\sum_{j=1}^{q} V_j/\theta_j}$ is

$$F_1(y; \boldsymbol{\eta}, \boldsymbol{\theta}, a = 0) = 1 - \sum_{j=1}^{p} \left(\prod_{\substack{i=1 \\ i \neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j} \right) \left(\prod_{i=1}^{q} \frac{\theta_i}{\theta_i + y\eta_j} \right), \quad y > 0.$$

Writing $a = 0$ is justified because this cdf arises from the cdf in (a) by simply setting $a = 0$.

(c) The cdf of $\dfrac{\sum_{j=1}^{p} U_j/\eta_j}{W}$ is

$$F_1(y; \boldsymbol{\eta}, \theta = \varnothing, a) = 1 - \sum_{j=1}^{p} \left(\prod_{\substack{i=1 \\ i \neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j} \right) \frac{1}{(1 + y\eta_j)^a}, \quad y > 0.$$

Here, $\boldsymbol{\theta} = \varnothing$ means that there are no θ's at all. As we can see, this cdf arises from the cdf in (a) by just omitting the product involving them.

Proposition 4. *Let* $V_1, \dots, V_p, U_1, \dots, U_q, Z_1, \dots, Z_h \overset{\text{iid}}{\sim} \mathcal{E}(1)$, $W \sim \mathcal{G}(a, 1)$ *be independent random variables with* $a > 0$. *Let also* $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_p)$, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_q)$, $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_h)$, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_h)$ *be vectors of positive numbers such that the* η*'s are distinct, the* θ*'s are also distinct and the ratio* λ_i/μ_i *is strictly increasing with respect to* i. *Set* $A_2 = \{y : \lambda_i + y\mu_i = 0 \text{ for some } i\}$. *For* $y \in \mathbb{R}\backslash A_2$ *let* $\beta_i = 1/(\lambda_i + y\mu_i)$ *for* $i = 1, \dots, h$ *and* θ_{i-h} *for* $i = h+1, \dots, h+q$. *Then we have the following:*

(a) The cdf of the random variable $Y = \dfrac{\sum_{i=1}^p V_i/\eta_i - \sum_{i=1}^q U_i/\theta_i - \sum_{i=1}^h \lambda_i Z_i}{\sum_{i=1}^h \mu_i Z_i + W}$ *is*

$$F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a) = \left\{1 - \sum_{j=1}^{p} \left(\prod_{\substack{i=1 \\ i \neq j}}^{p} \frac{\eta_i}{\eta_i - \eta_j}\right)\left(\prod_{i=1}^{q+h} \frac{\beta_i}{\beta_i + \eta_j}\right) \frac{1}{(1 + y\eta_j)^a}\right\} I_{(0,\infty)}(y) +$$

$$\sum_{\ell=0}^{h} \sum_{j=\ell+1}^{q+h} \left(\prod_{\substack{i=1 \\ i \neq j}}^{q+h} \frac{\beta_i}{\beta_i - \beta_j}\right)\left(\prod_{i=1}^{p} \frac{\eta_i}{\eta_i + \beta_j}\right) \frac{1}{(1 - y\beta_j)^a} I_{(-\rho_{\ell+1}, -\rho_\ell]}(y),$$

where $\rho_0 = 0$, $\rho_{h+1} = \infty$, *and* $\rho_i = \lambda_i/\mu_i$, $i = 1, \dots, h$.

(b) The cdf of $\dfrac{\sum_{i=1}^p V_i/\eta_i - \sum_{i=1}^h \lambda_i Z_i}{\sum_{i=1}^h \mu_i Z_i + W}$ *is* $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta} = \varnothing, \boldsymbol{\lambda}, \boldsymbol{\mu}, a)$. *Here again* $\boldsymbol{\theta} = \varnothing$ *means that there are no* θ*'s. This in turn implies that* $q = 0$ *and* $\beta_i = 1/(\lambda_i + y\mu_i)$ *for* $i = 1, \dots, h$.

(c) The cdf of $\dfrac{\sum_{i=1}^p V_i/\eta_i - \sum_{i=1}^q U_i/\theta_i - \sum_{i=1}^h \lambda_i Z_i}{\sum_{i=1}^h \mu_i Z_i}$ *is* $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a = 0)$.

(d) The cdf of $\dfrac{\sum_{i=1}^p V_i/\eta_i - \sum_{i=1}^q U_i/\theta_i}{W}$ *is* $F_2(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda} = \varnothing, \boldsymbol{\mu} = \varnothing, a)$. *This means that* $h = 0$ *and thus,* $\beta_i = \theta_i$, $i = 1, \dots, q$.

Proposition 5. *Let* $U_1, \dots, U_p, Z_1, \cdots, Z_h \overset{\text{iid}}{\sim} \mathcal{E}(1)$, $V_1, \dots, V_q \overset{\text{iid}}{\sim} \mathcal{G}(2, 1)$, $W \sim \mathcal{G}(a, 1)$ *(with* $a > 0$*) be independent random variables. Let also* $\boldsymbol{\eta} = (\eta_1, \dots, \eta_p)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_q)$, $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_h)$ *and* $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_h)$ *be vectors of positive numbers such that all* η*'s and* θ*'s are distinct and the ratio* λ_i/μ_i *is strictly increasing with respect to* i. *Set* $A_3 = \{y : \lambda_i - y\mu_i = 0 \text{ for some } i\}$ *and recall the definition of* ρ_i*'s from Proposition 4. For* $y \in (0, \infty)\backslash A_3$ *let* $\gamma_i = 1/(\lambda_i - y\mu_i)$ *for* $i = 1, \dots, h$ *and* η_{i-h} *for* $i = h+1, \dots, h+p$. *Then, for* $y > 0$ *we have the following:*

(a) The cdf of the random variable $Y = \dfrac{\sum_{i=1}^p U_i/\eta_i + \sum_{i=1}^q V_i/\theta_i + \sum_{i=1}^h \lambda_i Z_i}{\sum_{i=1}^h \mu_i Z_i + W}$ *is given by*

$$F_3(y; \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{\lambda}, \boldsymbol{\mu}, a) = 1 - \sum_{\ell=0}^{h} \sum_{j=\ell+1}^{h+p} \left(\prod_{\substack{i=1 \\ i \neq j}}^{h+p} \frac{\gamma_i}{\gamma_i - \gamma_j}\right)\left(\prod_{i=1}^{q} \frac{\theta_i}{\theta_i - \gamma_j}\right)^2 \frac{1}{(1 + y\gamma_j)^a} I_{[\rho_\ell, \rho_{\ell+1})}(y) -$$

$$\sum_{j=1}^{q}\left(\prod_{i=1}^{h+p}\frac{\gamma_i}{\gamma_i-\theta_j}\right)\left(\prod_{\substack{i=1\\i\neq j}}^{q}\frac{\theta_i}{\theta_i-\theta_j}\right)^2\frac{1}{(1+y\theta_j)^a}\left\{1+\frac{ay\theta_j}{1+y\theta_j}-\sum_{i=1}^{h+p}\frac{\theta_j}{\gamma_i-\theta_j}-\sum_{\substack{i=1\\i\neq j}}^{q}\frac{2\theta_j}{\theta_i-\theta_j}\right\}.$$

(b) The cdf of $Y=\dfrac{\sum_{i=1}^p U_i/\eta_i+\sum_{i=1}^h\lambda_i Z_i}{\sum_{i=1}^h\mu_i Z_i+W}$ *is* $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$*. Here again* $\boldsymbol{\theta}=\varnothing$ *means that there are no* θ*'s.*

(c) The cdf of $Y=\dfrac{\sum_{i=1}^p U_i/\eta_i+\sum_{i=1}^q V_i/\theta_i+\sum_{i=1}^h\lambda_i Z_i}{\sum_{i=1}^h\mu_i Z_i}$ *is* $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a=0)$*.*

(d) The cdf of $\dfrac{\sum_{i=1}^p U_i/\eta_i+\sum_{i=1}^q V_i/\theta_i}{W}$ *is* $F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda}=\varnothing,\boldsymbol{\mu}=\varnothing,a)$*. This means that* $h=0$ *and thus,* $\gamma_i=\eta_i$*,* $i=1,\ldots,p$*, and* $\rho_1=\infty$*.*

Proposition 6. *Let* $V_1,\cdots,V_p,Z_1,Z_2\overset{\text{iid}}{\sim}\mathcal{E}(1)$*,* $W\sim\mathcal{G}(a,1)$ *(with* $a>0$*) be independent random variables. Further, let* $\boldsymbol{\eta}=(\eta_1,\cdots,\eta_p)$*,* λ_1,λ_2 *be vectors of positive numbers such that* η*'s are distinct. Then the cdf of the random variable* $\dfrac{\sum_{i=1}^p U_i/\eta_i-\lambda_1 Z_1+\lambda_2 Z_2}{Z_1+Z_2+W}$ *is*

$$F_4(y;\boldsymbol{\eta},\boldsymbol{\lambda},a)=\left\{1-\sum_{j=1}^{p+I_{(0,\lambda_2]}(y)}\left(\prod_{\substack{i=1\\i\neq j}}^{p+2}\frac{\zeta_i}{\zeta_i-\zeta_j}\right)\frac{1}{(1+y\zeta_j)^a}\right\}I_{(0,\infty)}(y)+$$
$$\left(\prod_{i=1}^{p+1}\frac{\zeta_i}{\zeta_i-\zeta_{p+2}}\right)\frac{1}{(1+y\zeta_{p+2})^a}I_{(-\lambda_1,0]}(y),$$

where $\zeta_i=\eta_i$*,* $i=1\ldots,p$*,* $\zeta_{p+1}=1/(\lambda_2-y)$*,* $\zeta_{p+2}=-1/(\lambda_1+y)$*.*

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Case $\max(r,s) < m$			
Range of d	Conditional distribution of T_1 given $D=d$	Range of d	Conditional distribution of T_1 given $D=d$
$[0,r]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=m-r-1$, $a=n-m-s,\ \boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=((n-r-1)/(r+1),\ldots,(n-m+1)/(m-1))$	$(m,n-s)$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=d-[n/2]$, $a=2n-m-r-s-d,\ \boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=(d/(n-d),\ldots,(n-m+1)/(m-1))$
(r,m)	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=m-d$, $a=n-m-s-r+d,\ \boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=((n-d)/d,\ldots,(n-m+1)/(m-1))$,	$[n-s,n-s+k)$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k+1,\ q=m-s-1$, $a=n-m-r,\ \boldsymbol{\eta}=(s-k+1,\ldots,n-d,n-s,\ldots,d)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\ldots,(n-m+1)/(m-1))$
$m\ (\neq n-s)$	If $n=2m-1,\ F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,q=1$, $a=n-s-r-1,\ \boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=(m/(m-1))$ while if $n=2m,\ F_1(y;\boldsymbol{\eta},\theta=\varnothing,a)$, with $p=k$, $a=n-s-r,\ \boldsymbol{\eta}=(s-k+1,\ldots,s)$	$[n-s+k,n]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=m-s-1$, $a=n-m-r,\ \boldsymbol{\eta}=(n-s,\ldots,n-s+k-1)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\ldots,(n-m+1)/(m-1))$

Case $s \geqslant m$		Case $r \geqslant m$	
Range of d	Conditional distribution of T_1 given $D=d$	Range of d	Conditional distribution of T_1 given $D=d$
$[0,r]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a=0)$ with $p=k,\ q=n-r-s-1$, $\boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=((n-r-1)/(r+1),\ldots,(s+1)/(n-s-1))$.	$[0,r]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$ with $p=k,\ a=n-r-s-1$ $\boldsymbol{\eta}=(s-k+1,\ldots,s)$
$(r,n-s)$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=n-s-d,\ a=d-r$, $\boldsymbol{\eta}=(s-k+1,\ldots,s)$, $\boldsymbol{\theta}=((n-d)/d,\ldots,(s+1)/(n-s-1)$	$(r,n-s)$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a)$ with $p=k,\ q=d-r$, $a=n-s-d,\ \boldsymbol{\eta}=(s,\ldots,s-k+1)$, $\boldsymbol{\theta}=(d/(n-d),\ldots,(r+1)//(n-r-1))$
$[n-s,n-s+k)$	Let $\boldsymbol{\eta}$ be the vector consisting of the distinct entries of $(s-k+1,\ldots,n-d,n-s,\ldots,d)$ and $\boldsymbol{\theta}$ the $q-$dimensional vector of its tied entries (pairs) each taken once. If $q=0,\ F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$; if $q>0,\ F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda}=\varnothing,\boldsymbol{\mu}=\varnothing,a)$. In both cases $p=k+1-2q,\ a=n-r-s-1$	$[n-s,n-s+k)$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a=0)$ with $p=k+1,\ q=n-r-s-1$, $\boldsymbol{\eta}=(s-k+1,\ldots,n-d,n-s,\ldots,d)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\ldots,(r+1)/(n-r-1))$
$[n-s+k,n]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,a)$ with $p=k,\ a=n-r-s-1$, $\boldsymbol{\eta}=(n-s,\ldots,n-s+k-1)$	$[n-s+k,n]$	$F_1(y;\boldsymbol{\eta},\boldsymbol{\theta},a=0)$ with $p=k,\ q=n-r-s-1$, $\boldsymbol{\eta}=(n-s,\ldots,n-s+k-1)$, $\boldsymbol{\theta}=((n-s-1)/(s+1),\ldots,(r+1)/(n-r-1))$

Table 1: List of the conditional distributions of $T_1(n,r,s,k)/(n-r-s)$ given $D=d$. The distribution functions $F_1,\ F_3$ are given in Propositions 3 and 5 in the Appendix.

Case $\boldsymbol{\max(r,s) < m}$		
Range of d	Range of d'	Conditional distribution of $T_3/(n-r-s)$ given $D=d,\ D'=d'$
$[0,r]$	$[0,k)$	$F_2(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$, with $p=k-d',\ q=r-d+1,\ a=m-s-1,\ \boldsymbol{\eta}=(n'-k+1,\ldots,n'-d'),\ \boldsymbol{\theta}=(n-r,\ldots,n-d)$. If $n=2m-1,\ h=m-r-1,\ \boldsymbol{\lambda}=(1/m,\ldots,1/(n-r-1)),\ \boldsymbol{\mu}=((m-1)/m,\ldots,(r+1)/(n-r-1))$, while if $n=2m,\ h=m-r,\ \boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/(n-r-1)),\ \boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,(r+1)/(n-r-1))$.
	$[k,n']$	$1-F_3(-y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $\boldsymbol{\eta},\ \boldsymbol{\theta}$ the vectors of untied and tied entries of $(k,\ldots,d',n-r,\ldots,n-d)$, respectively, $q=\dim(\boldsymbol{\theta}),\ p=d'-k+r-d+2-2q$, and $a,h,\ \boldsymbol{\lambda},\ \boldsymbol{\mu}$ as in the previous case.
(r,m)	$[0,k)$	$F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d',\ a=m-r-s+d-1,\ \boldsymbol{\eta}=(n'-k+1,\ldots,n'-d')$. If $n=2m-1,\ h=m-d,\ \boldsymbol{\lambda}=(1/m,\ldots,1/(n-d)),\ \boldsymbol{\mu}=((m-1)/m,\ldots,d/(n-d))$; if $n=2m,\ h=m-d+1,\ \boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/(n-d)),\ \boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,d/(n-d))$.
	$[k,n']$	$1-F_3(-y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k,\ \boldsymbol{\eta}=(k,\ldots,d')$ and $a,\ h,\ \boldsymbol{\lambda},\ \boldsymbol{\mu}$ as above.
$m\ (\neq n-s)$	$[0,k)$	If $n=2m-1,\ F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d',\ \boldsymbol{\eta}=(n'-k+1,\ldots,n'-d'),\ a=n-r-s-1,\ h=1,$ $\boldsymbol{\lambda}=(1/m),\ \boldsymbol{\mu}=((m-1)/m)$; if $n=2m,\ F_4(y;\boldsymbol{\eta},\boldsymbol{\lambda},a)$, with same $p,\ \boldsymbol{\eta}$ and $a=n-r-s-2,\ \boldsymbol{\lambda}=(1/2,1/2)$.
	$[k,n']$	If $n=2m-1,\ 1-F_2(-y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k,\ \boldsymbol{\eta}=(k,\ldots,d'),\ a=n-r-s-1,\ h=1,\ \boldsymbol{\lambda}=(1/m),$ $\boldsymbol{\mu}=((m-1)/m)$; *if* $n=2m,\ 1-F_4(-y;\boldsymbol{\eta},\boldsymbol{\lambda},a)$ with the same $p,\ \boldsymbol{\eta}$, and $a=n-r-s-2,\ \boldsymbol{\lambda}=(1/2,1/2)$.
$(m,n-s)$	$[0,k)$	$F_3(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d',\ h=d-m+1,\ \boldsymbol{\eta}=(n'-k+1,\ldots,n'-d'),\ a=n-r-s-d+m-1$. If $n=2m-1,\ \boldsymbol{\lambda}=(1/m,\ldots,1/d),\boldsymbol{\mu}=((m-1)/m,\ldots,(n-d)/d)$; if $n=2m,\ \boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/d),\ \boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,(n-d)/d)$.
	$[k,n']$	$1-F_2(-y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k,\ h=d-m+1,\ \boldsymbol{\eta}=(k,\ldots,d'),\ a=n-r-s-d+m-1$ and $h,\ \boldsymbol{\lambda},\ \boldsymbol{\mu}$ as above.
$[n-s,n]$	$[0,k)$	$F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$, with $\boldsymbol{\eta},\ \boldsymbol{\theta}$ the vectors of untied and tied entries of $(n'-k+1,\ldots,n'-d',n-s,\ldots,d)$, respectively, $q=\dim(\boldsymbol{\theta}),\ p=k-d'+d-n+s+1-2q,\ a=m-r-1$. If $n=2m-1,\ h=m-s-1,\ \boldsymbol{\lambda}=(1/m,\ldots,1/(n-s-1)),\ \boldsymbol{\mu}=((m-1)/m,\ldots,(s+1)/(n-s-1))$; if $n=2m,\ h=m-s,\ \boldsymbol{\lambda}=(1/2m,1/(m+1),\ldots,1/(n-s-1))\boldsymbol{\mu}=(1,(m-1)/(m+1),\ldots,(s+1)/(n-s-1))$.
	$[k,n']$	$1-F_2(-y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k,\ q=d-n+s+1,\ a=m-r-1,\ \boldsymbol{\eta}=(k,\ldots,d'),\ \boldsymbol{\theta}=(n-s,\ldots,d)$ and $h,\ \boldsymbol{\lambda},\ \boldsymbol{\mu}$ as above.

Table 2: List of the conditional distributions of $T_3(n,r,s,n',k)/(n-r-s)$ given $D=d,\ D'=d'$. The distribution functions $F_2,\ F_3,\ F_4$ are given in Propositions 4, 5, 6 in the Appendix.

Case $\boldsymbol{s \geqslant m}$		
Range of d	Range of d'	Conditional distribution of $T_3^*/(n-r-s)$ given $D=d$, $D'=d'$
$[0,r]$	$[0,k)$	$F_2(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a=0)$, with $p=k-d'$, $q=r-d+1$, $\boldsymbol{\eta}=(n'-k+1,\dots,n'-d')$, $\boldsymbol{\theta}=(n-r,\dots,n-d)$, $h=n-r-s-1$, $\boldsymbol{\lambda}=(1/(n-r-1),\dots,1/(s+1))$, $\boldsymbol{\mu}=((r+1)/(n-r-1),\dots,(n-s-1)/(s+1))$
	$[k,n']$	$1-F_3(-y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda},\boldsymbol{\mu},a=0)$, with $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ the vectors of untied and tied entries of $(k,\dots,d',n-r,\dots,n-d)$, respectively, $q=\dim(\boldsymbol{\theta})$, $p=d'-k+r-d+2-2q$, and h, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ as above.
$(r,n-s)$	$[0,k)$	$F_2(y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=k-d'$, $a=d-r$, $\boldsymbol{\eta}=(n'-k+1,\dots,n'-d')$, $h=n-s-d$, $\boldsymbol{\lambda}=(1/(s+1),\dots,1/(n-d))$, $\boldsymbol{\mu}=((n-s-1)/(s+1),\dots,d/(n-d))$
	$[k,n']$	$1-F_3(-y;\boldsymbol{\eta},\boldsymbol{\theta}=\varnothing,\boldsymbol{\lambda},\boldsymbol{\mu},a)$ with $p=d'+1-k$, $\boldsymbol{\eta}=(k,\dots,d')$ and a, h, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ as above.
$[n-s,n]$	$[0,k)$	$F_3(y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda}=\varnothing,\boldsymbol{\mu}=\varnothing,a)$ with $\boldsymbol{\eta}$, $\boldsymbol{\theta}$ the vectors of untied and tied entries of $(n'-k+1,\dots,n'-d',n-s,\dots,d)$, respectively, $q=\dim(\boldsymbol{\theta})$, $p=k-d'+d-n+s+1-2q$, $a=n-r-s-1$
	$[k,n']$	$1-F_2(-y;\boldsymbol{\eta},\boldsymbol{\theta},\boldsymbol{\lambda}=\varnothing,\boldsymbol{\mu}=\varnothing,a)$ with $p=d'-k+1$, $q=d-n+s+1$, $a=n-r-s-1$, $\boldsymbol{\eta}=(k,\dots,d')$, $\boldsymbol{\theta}=(n-s,\dots,d)$

Table 3: List of the conditional distributions of $T_3^*(n,r,s,n',k)/(n-r-s)$ given $D=d$, $D'=d'$. The distribution functions F_2, F_3, F_4 are given in Propositions 4, 5, 6 in the Appendix.

n	r	s	k	0.975	0.95	0.05	0.025	Estimated cdf			
5	0	1	1	0.032306	0.065586	6.013345	8.390094	.0250	.0497	.9497	.9750
5	0	2	1	0.022086	0.044874	5.268485	7.989312	.0249	.0502	.9498	.9749
5	0	2	2	0.236190	0.352240	13.404938	19.998340	.0250	.0502	.9497	.9749
5	0	3	1	0.031686	0.064699	21.076159	43.162979	.0253	.0502	.9505	.9753
5	0	3	3	0.604059	0.844875	79.488683	161.390983	.0251	.0501	.9502	.9752
5	1	2	1	0.027529	0.056253	17.776676	36.325196	.0249	.0501	.9500	.9750
5	1	2	2	0.266836	0.405444	48.870984	99.305400	.0250	.0500	.9497	.9748
5	2	1	1	0.046336	0.094894	30.912357	63.118980	.0250	.0501	.9498	.9750
10	0	1	1	0.027046	0.054875	3.835644	4.935865	.0253	.0501	.9501	.9752
10	0	2	1	0.013752	0.027902	1.990371	2.577785	.0249	.0498	.9500	.9751
10	0	2	2	0.177196	0.262413	5.068413	6.372565	.0249	.0501	.9503	.9753
10	0	5	1	0.008137	0.016515	1.332610	1.799080	.0249	.0496	.9496	.9748
10	0	5	5	0.688290	0.868568	9.834964	12.706394	.0248	.5000	.9500	.9752
10	1	5	1	0.008595	0.017465	1.624275	2.278713	.0249	.0497	.9500	.9746
10	1	5	5	0.691779	0.880539	12.424051	16.763694	.0249	.0500	.9500	.9752
10	3	1	1	0.028090	0.057045	4.450853	5.892681	.0253	.0502	.9502	.9751
10	3	5	1	0.012284	0.025167	8.590256	17.606603	.0250	.0499	.9497	.9748
10	3	5	5	0.754130	1.007138	81.085102	164.632174	.0249	.0499	.9499	.9750
15	0	5	1	0.005561	0.011278	0.762622	0.975696	.0248	.0499	.9503	.9752
15	0	5	5	0.627788	0.782190	6.188777	7.477179	.0250	.0501	.9497	.9747
15	0	10	1	0.006543	0.013292	1.127923	1.531165	.0248	.0497	.9498	.9748
15	0	10	10	1.231010	1.480241	12.893970	16.513694	.0251	.0500	.9500	.9749
15	2	10	5	0.301057	0.385097	7.797347	11.552998	.0252	.0503	.9499	.9748
15	2	10	10	1.235579	1.530926	27.810869	41.151330	.0252	.0500	.9499	.9748
15	10	2	2	0.222144	0.335071	14.413208	21.718113	.0248	.0498	.9499	.9749

Table 4: Left: Upper α-quantiles of $T_1(n, r, s, k)$ used in 95% prediction intervals and bounds for $X_{n-s+k:n}$ for selected configurations (n, r, s, k). Right: Estimated cdf of $T_1(n, r, s, k)$ at the correspoding quantiles based on 10^6 simulations.

3.12	17.63	17.64	19.56	19.74	20.32	20.82	22.63
23.02	23.03	23.06	23.36	24.81	27.67	38.75	

Table 5: Simulated dataset from $\mathcal{L}(20, 5)$ used in Example 6.1.

k	$\alpha = .975$	$\alpha = .025$	k	95% p.i.	k	95% p.i.
1	-6.6748	-1.0072	1	$[-12.303, 6.745]$	11	$[8.041, 12.735]$
2	-4.1939	-0.6115	2	$[-3.965, 8.075]$	12	$[8.502, 13.323]$
3	-3.1369	-0.3615	3	$[-0.413, 8.915]$	13	$[8.925, 14.001]$
4	-2.4885	-0.1731	4	$[1.766, 9.548]$	14	$[9.331, 14.799]$
5	-2.0279	-0.0190	5	$[3.314, 10.066]$	15	$[9.744, 15.757]$
6	-1.6742	0.1148	6	$[4.503, 10.516]$	16	$[10.194, 16.946]$
7	-1.3891	0.2378	7	$[5.461, 10.929]$	17	$[10.712, 18.494]$
8	-1.1519	0.3586	8	$[6.259, 11.335]$	18	$[11.345, 20.673]$
9	-0.9499	0.4843	9	$[6.937, 11.758]$	19	$[12.185, 24.225]$
10	-0.7750	0.6214	10	$[7.525, 12.219]$	20	$[13.515, 32.563]$

Table 6: The .975- and .025-upper quantiles of $T_3(33, 0, 0, 20, k)$, $k = 1, \ldots, 10$, and the corresponding prediction intervals for $Y_{k:20}$, $k = 1, \ldots, 20$; see Section 6.2.