

INTERVAL ESTIMATION FOR THE RATIO OF SCALE PARAMETERS AND FOR ORDERED SCALE PARAMETERS ¹

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Abstract

Confidence intervals for the ratio of scale parameters are constructed in general families of distributions with nuisance (location) parameters. Each of these intervals has coverage probability at least as large as that of the standard minimum size (i.e., minimum ratio of endpoints) interval and, in addition, smaller size. Then analogous improved confidence intervals for the scale parameters subject to order restriction are derived. The method of construction is similar to that in Goutis and Casella [5], [6]. Examples are given and include the normal and exponential distributions as well as the inverse Gaussian distribution which is not a purely location–scale model. Applications to interval estimation of the error variance in variance components models are also discussed.

1 Introduction

There are many situations where data (measurements) are available from two sources for the same objective. For example, an experiment may be conducted by two laboratories using different methods or different measuring instruments. To effectively understand whether there is a difference in variability between the two sources, both point and interval estimates of the ratio of the corresponding population variances are required. Furthermore, if the two populations are assumed to be normal, a confidence interval for the ratio of variances that contains one may indicate that a two sample t-test is applicable for testing equality of the respective means.

Nagata [23] derived intervals for the ratio of the variances of two normal populations that were better than the standard F-interval, by extending arguments of Stein [25]. In

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Section 2, under suitable monotone likelihood ratio conditions, confidence intervals for the ratio σ_1/σ_2 of the scale parameters σ_1 and σ_2 of two independent populations with nuisance (location) parameters are constructed, which dominate the standard minimum ratio of endpoints interval in terms of both coverage probability and ratio of endpoints. The construction is based on a modification of Goutis and Casella's [5], [6] technique. Interestingly, in some typical cases, the computation of the improved intervals involves exactly the same algorithm that produces the standard interval (but with different input variables), see Remark 2.1. Thus, the new intervals are easy to use in practice. Examples are given and include the normal, exponential, and inverse Gaussian distributions. Numerical results are also presented to indicate the magnitude of the improvement over the standard interval.

We mention that the companion problem of (decision theoretic) point estimation of σ_1/σ_2 has been studied by several researchers, see Gelfand and Dey [3], Madi and Tsui [20], Kubokawa [13], Madi [19], Ghosh and Kundu [4], Kubokawa and Srivastava [16].

In Section 3, interval estimation of σ_1 subject to the restriction that $\sigma_1 \leq \sigma_2$ is considered. This restriction arises very naturally in several problems of practical interest. For instance, suppose that the lifetime of a component manufactured using an old method follows the exponential distribution with hazard rate σ_2 and the lifetime of the same component manufactured using a new technology follows the exponential distribution with hazard rate σ_1 . It is then reasonable to expect that $\sigma_1 \leq \sigma_2$. Using the same monotone likelihood ratio conditions and exploiting again Goutis and Casella's [5], [6] idea, confidence intervals for σ_1 are constructed which improve on the standard intervals, i.e., those that do not take into account the order restriction. As before, the improvement is both in coverage probability and ratio of endpoints. It is noted that in typical cases the computational difficulty of the improved intervals is essentially the same as that of their standard counterparts, see Remark 3.1. Thus, the new intervals share practical merit. The results apply, in particular, to the problem of interval estimation of the error variance in a balanced one way random effects model, and (for this model) the magnitude of the improvement over the standard interval is investigated numerically. Interval estimation of σ_2 subject to $\sigma_1 \leq \sigma_2$ is also outlined.

We note that the book by Robertson, Wright, and Dykstra [24] contains a detailed account on order restricted inference. Early work on this subject traces back to van Eeden [26], [27], [28], [29], [30] and recent developments are contained in van Eeden [31]. Relevant work includes Kushary and Cohen [17], Kubokawa and Saleh [14], Hwang and Peddada [7], and Cohen and Sackrowitz [2].

An appendix, Section 4, contains some technical results. We close this section by mentioning that the technique of Goutis and Casella [5], [6] was also used by Iliopoulos and Kourouklis [9] for the construction of an improved interval for the generalized variance in a multivariate normal context.

2 Improved intervals for the ratio σ_1/σ_2

Let S_1, S_2, T_1, T_2 be independent statistics such that S_i/σ_i and T_i/σ_i have densities

$$g_i(x)I_{(0,\infty)}(x) \quad \text{and} \quad h_i(x; \mu_i, \sigma_i)I_{(\lambda_i,\infty)}(x)$$

respectively, where μ_i, σ_i are unknown parameters, σ_i is positive and $\lambda_i = \lambda_i(\mu_i, \sigma_i)$ is a real valued function of μ_i and σ_i , $i = 1, 2$. The problem is to construct a ‘‘good’’ confidence interval for the ratio $\sigma = \sigma_1/\sigma_2$. For instance, if X_1, \dots, X_n and Y_1, \dots, Y_m are two independent random samples from exponential distributions $E(\mu_1, \sigma_1)$ and $E(\mu_2, \sigma_2)$ then $S_1 = \sum_{i=1}^n (X_i - X_{(1)})$, $S_2 = \sum_{i=1}^m (Y_i - Y_{(1)})$, $T_1 = X_{(1)} = \min\{X_i : i = 1, \dots, n\}$, $T_2 = Y_{(1)} = \min\{Y_i : i = 1, \dots, m\}$ and $\lambda_i = \mu_i/\sigma_i$. See also Kubokawa [13] regarding the above framework.

We require the following condition.

(A1) $\frac{g_i(c_1x)}{g_i(c_2x)}$ is strictly increasing in $x > 0$ for $0 < c_1 < c_2$, $i = 1, 2$.

We set $S = S_1/S_2$ and denote by $g(x), x > 0$, the density of S/σ . Because of (A1), it follows from Lemmas 4.1 and 4.4 in the appendix that $g(x)$ is continuous and $xg(x)$ is unimodal. The standard confidence intervals for σ have the form $I = (S/b, S/a)$, $0 < a < b$, and among them the one with confidence coefficient $1 - \alpha$ and minimum ratio of endpoints is given by

$$I_0 = (S/b, S/a),$$

where a and b uniquely satisfy

$$\int_a^b g(x)dx = 1 - \alpha \quad \text{and} \quad ag(a) = bg(b). \quad (2.1)$$

We will need some further assumptions and a preliminary result. We suppose that there is a value (μ_{10}, σ_{10}) of (μ_1, σ_1) with $\lambda_1(\mu_{10}, \sigma_{10}) = 0$ such that setting $h_1(x) = h_1(x; \mu_{10}, \sigma_{10}), x > 0$, the following two conditions hold.

(A2) $\frac{h_1(x; \mu_1, \sigma_1)I_{(\lambda_1,\infty)}(x)}{h_1(x)}$ is nondecreasing in $x > 0$.

(A3) $\frac{h_1(c_1x)}{h_1(c_2x)}$ is strictly increasing in $x > 0$ for $0 < c_1 < c_2$.

Let $W_1 = T_1/S_1$. Then, given $W_1 = w_1 > 0$, the conditional density of S/σ is

$$f_1(x|w_1; \mu_1, \sigma_1) \propto \int_0^\infty xv^2 g_1(xv) h_1(xvw_1; \mu_1, \sigma_1) g_2(v) I_{(\max(0, \lambda_1), \infty)}(xvw_1) dv. \quad (2.2)$$

For the sake of simplicity, when $\mu_1 = \mu_{10}$ and $\sigma_1 = \sigma_{10}$ in (2.2) we write $f_1(x|w_1)$, that is

$$f_1(x|w_1) \propto \int_0^\infty xv^2 g_1(xv) h_1(xvw_1) g_2(v) dv. \quad (2.3)$$

The following result is an application of Lemmas 4.1, 4.3, and 4.4.

Lemma 2.1 (i) For $w_1 > 0$, under (A1) and (A3), we have that $f_1(x|w_1)$ is a continuous function of $x > 0$, $xf_1(x|w_1)$ is a unimodal function of $x > 0$, and $\lim_{x \rightarrow 0} xf_1(x|w_1) = \lim_{x \rightarrow \infty} xf_1(x|w_1) = 0$.
(ii) For $w_1 > 0$, under (A1) and (A2), $f_1(x|w_1; \mu_1, \sigma_1)/f_1(x|w_1)$ is strictly increasing in $x > 0$ unless the ratio in (A2) is constant as a function of $x > 0$.

We now seek to improve on I_0 by constructing another interval that has smaller ratio of endpoints while maintaining the coverage probability above the nominal level $1 - \alpha$. To motivate our choice of improved interval, we note that in typical cases (as, for instance, in Examples 1–3 below) the problem of interval estimation of σ is invariant under the group of scale transformations $(T_1, S_1, T_2, S_2) \rightarrow (c_1T_1, c_1S_1, c_2T_2, c_2S_2)$, $c_i > 0$, $i = 1, 2$, and the corresponding class of scale equivariant intervals is given by $\{(\psi_1(W_1, W_2)S, \psi_2(W_1, W_2)S) : \psi_1 < \psi_2 \text{ positive functions}\}$, where $W_2 = T_2/S_2$. A subclass which is simpler to study consists of intervals of the form

$$I(S, W_1) = (\phi_1(W_1)S, \phi_2(W_1)S) ,$$

for positive functions $\phi_1 < \phi_2$. It is the latter class that we will next consider. Following Goutis and Casella [5], [6] we will require that $I(S, W_1)$ and I_0 have the same coverage probability when $\mu_1 = \mu_{10}$ and $\sigma_1 = \sigma_{10}$. Using Stein's [25] idea, this requirement will hold if it holds conditionally for each given value of W_1 . Since we also seek to improve on the ratio of endpoints we will in addition require that ϕ_2/ϕ_1 is minimized. To this end, for each $w_1 > 0$, we determine $\phi_{10} = \phi_{10}(w_1)$ and $\phi_{20} = \phi_{20}(w_1)$ from the equations

$$\int_{1/\phi_{20}}^{1/\phi_{10}} f_1(x|w_1)dx = \int_a^b f_1(x|w_1)dx \text{ and } \frac{1}{\phi_{10}}f_1\left(\frac{1}{\phi_{10}}|w_1\right) = \frac{1}{\phi_{20}}f_1\left(\frac{1}{\phi_{20}}|w_1\right) . \quad (2.4)$$

The existence and uniqueness of ϕ_{10} and ϕ_{20} follows from Lemma 2.1(i). Define now a confidence interval for σ by

$$I_1(S, W_1) = (\phi_1(W_1)S, \phi_2(W_1)S) ,$$

where

$$\phi_1(w_1) = \begin{cases} \min\{\phi_{10}(w_1), 1/b\} , & \text{if } w_1 > 0 \\ 1/b & , \text{if } w_1 \leq 0 \end{cases} \quad \text{and} \quad \phi_2(w_1) = \begin{cases} \min\{\phi_{20}(w_1), 1/a\} , & \text{if } w_1 > 0 \\ 1/a & , \text{if } w_1 \leq 0 . \end{cases} \quad (2.5)$$

The next theorem establishes the superiority of $I_1(S, W_1)$ over I_0 . Condition (2.6) below is exemplified in Theorem 2.3. Also, the underlying value of (μ_1, σ_1) in the probability statements of the theorem is that specified in (A2).

Theorem 2.2 Assume that (A1), (A2), and (A3) hold. If

$$P(W_1 > 0 \text{ and } bf_1(b|W_1) > af_1(a|W_1)) > 0 \quad (2.6)$$

then we have the following.

(i) $P(\sigma \in I_1(S, W_1)) \geq P(\sigma \in I_0) = 1 - \alpha$ and the inequality is strict unless the ratio in (A2) is constant as a function of $x > 0$.

(ii) The ratio of endpoints of $I_1(S, W_1)$ is less than or equal to b/a (i.e., the ratio of endpoints of I_0) and is strictly less with positive probability.

Proof. (i) Let B the set on which $I_1(S, W_1)$ differs from I_0 , i.e., $B = \{w_1 : w_1 > 0 \text{ and } \phi_{10}(w_1) < 1/b\}$. We first show that $B = \{w_1 : w_1 > 0 \text{ and } bf_1(b|w_1) > af_1(a|w_1)\}$. Let $w_1 > 0$ be such that

$$bf_1(b|w_1) > af_1(a|w_1). \quad (2.7)$$

Trivially, the first equality in (2.4) implies that neither of the intervals $(1/\phi_{20}, 1/\phi_{10})$ and (a, b) can be a proper subset of the other. Consequently, using the second equality in (2.4), (2.7), and the unimodality of $xf_1(x|w_1)$ (Lemma 2.1 (i)), from Lemma 4.5 we conclude that $\phi_{10}(w_1) < 1/b$. Hence, $B \supseteq \{w_1 : w_1 > 0 \text{ and } bf_1(b|w_1) > af_1(a|w_1)\}$. Conversely, consider $w_1 > 0$ with $\phi_{10}(w_1) < 1/b$. Then $a < b \leq 1/\phi_{20} < 1/\phi_{10}$ or $a < 1/\phi_{20} \leq b < 1/\phi_{10}$. In either case, the unimodality of $xf_1(x|w_1)$ and (2.4) imply that $bf_1(b|w_1) > af_1(a|w_1)$, and thus $B \subseteq \{w_1 : w_1 > 0 \text{ and } bf_1(b|w_1) > af_1(a|w_1)\}$. To establish (i), since the two intervals differ only on the set B and $P(B) > 0$, by (2.6), it suffices to show that

$$P(\sigma \in I_1(S, W_1)|W_1 = w_1) \geq P(\sigma \in I_0|W_1 = w_1) \quad (2.8)$$

for each $w_1 \in B$ with strict inequality holding unless the ratio in (A2) is constant as a function of $x > 0$. In the case of constant ratio in (A2), $f_1(x|w_1; \mu_1, \sigma_1) = f_1(x|w_1)$ and hence we have equality in (2.8) for all $w_1 \in B$ because of (2.4). Suppose now that the ratio in (A2) is not constant and consider $w_1 \in B$. Then, (2.4), the relation $\phi_{10}(w_1) < 1/b$, and the fact that $f_1(x|w_1; \mu_1, \sigma_1)/f_1(x|w_1)$ is strictly increasing (Lemma 2.1 (ii)) imply $\int_{1/\phi_{20}(w_1)}^{1/\phi_{10}(w_1)} f_1(x|w_1; \mu_1, \sigma_1)dx > \int_a^b f_1(x|w_1; \mu_1, \sigma_1)dx$, by Lemma 4.6. Thus, (2.8) holds as strict inequality.

(ii) For a given value w_1 of W_1 , the ratio of endpoints of $I_1(S, W_1)$ is either b/a or ϕ_{20}/ϕ_{10} . In the latter case, $w_1 \in B$ and we have $\phi_{20}/\phi_{10} < b/a$ since by construction $y = 1/\phi_{10}$ and $t = 1/\phi_{20}$ uniquely minimize y/t subject to $\int_t^y f_1(x|w_1)dx = \delta$, where $\delta = \int_a^b f_1(x|w_1)dx$, see (2.4). The result now follows since $P(B) > 0$.

In a similar fashion we can construct an improved interval for σ of the form

$$I(S, W_2) = (\phi_1(W_2)S, \phi_2(W_2)S). \quad (2.9)$$

In this case, instead of (A2) and (A3), we need to suppose that there is a value (μ_{20}, σ_{20}) of (μ_2, σ_2) with $\lambda_2(\mu_{20}, \sigma_{20}) = 0$ such that setting $h_2(x) = h_2(x; \mu_{20}, \sigma_{20})$, $x > 0$, the following two conditions hold.

(A'2) $\frac{h_2(x; \mu_2, \sigma_2)I_{(\lambda_2, \infty)}(x)}{h_2(c_1x)}$ is nondecreasing in $x > 0$.

(A'3) $\frac{h_2(c_1x)}{h_2(c_2x)}$ is strictly increasing in $x > 0$ for $0 < c_1 < c_2$.

Details of the construction of $I(S, W_2)$ are given in Iliopoulos and Kourouklis [8].

The improved intervals in Theorem 2.2 and in (2.9) use only one of T_1 and T_2 . It would be of considerable interest to construct improved intervals that use both of these statistics. The conditions in Theorem 2.2 are illustrated in the following general result.

Theorem 2.3 *Suppose that $g_i(x)$ is a gamma density and that (μ_{10}, σ_{10}) in (A3) is such that $h_1(x)$ is also a gamma density, i.e., $g_i(x) \propto x^{\alpha_i-1}e^{-x/\beta_i}$ and $h_1(x) \propto x^{\gamma_1-1}e^{-x/\delta_1}$, $x > 0$, $\alpha_i, \beta_i, \gamma_1, \delta_1 > 0$, $i = 1, 2$. Then (A1), (A3), and (2.6) hold.*

Proof. (A1) and (A3) are straightforward to show. Thus we only prove (2.6). Using (2.3) and the assumption of the theorem, it is easily seen that, for $w_1 > 0$, $bf_1(b|w_1) > af_1(a|w_1)$ holds if and only if $w_1 < w_{10}$, where w_{10} is given by

$$w_{10} = \delta_1 \left\{ \frac{1}{\beta_2} \frac{b^d - a^d}{ba^d - ab^d} - \frac{1}{\beta_1} \right\} \quad (2.10)$$

and $d = (\alpha_1 + \gamma_1)/(\alpha_1 + \gamma_1 + \alpha_2)$. Next, evaluating and substituting $g(x)$ in the second equation of (2.1) gives

$$\frac{1/\beta_2 + b/\beta_1}{1/\beta_2 + a/\beta_1} = \left\{ \frac{b}{a} \right\}^{\frac{\alpha_1}{\alpha_1 + \alpha_2}}. \quad (2.11)$$

Thus, from (2.10) and (2.11) we obtain that $w_{10} > 0$, and consequently $\{w_1 > 0 : bf_1(b|w_1) > af_1(a|w_1)\} = \{w_1 : 0 < w_1 < w_{10}\}$. Since W_1 is supported on $(0, \infty)$ or $(-\infty, \infty)$ (depending on whether $\lambda_1 \geq 0$ or $\lambda_1 < 0$) (2.6) holds.

Remark 2.1. We discuss some computational aspects of the interval $I_1(S, W_1)$ for the case considered in Theorem 2.3. First recall that for the computation of I_0 one needs to solve

$$\int_a^b g(x; \alpha_1) dx = 1 - \alpha \quad \text{and} \quad ag(a; \alpha_1) = bg(b; \alpha_1)$$

for a and b . Here $g(x; \alpha_1)$ is the density of the ratio of two independent gamma densities, the notation stressing the dependency of the function g on the shape parameter α_1 of $g_1(x)$. Let s and w_1 be the observed values of S and W_1 . Then $I_1(s, w_1) \neq I_0$ only if $0 < w_1 < w_{10}$. In that case, it is not difficult to show that $f_1(x|w_1) \propto g((1 + w_1\beta_1/\delta_1)x; \alpha_1 + \gamma_1)$. Next, because of (2.4) and with $\epsilon = \int_{a(1+w_1\beta_1/\delta_1)}^{b(1+w_1\beta_1/\delta_1)} g(x; \alpha_1 + \gamma_1) dx$, to compute $I_1(s, w_1)$ one solves

$$\int_{a^*}^{b^*} g(x; \alpha_1 + \gamma_1) dx = \epsilon \quad \text{and} \quad a^*g(a^*; \alpha_1 + \gamma_1) = b^*g(b^*; \alpha_1 + \gamma_1),$$

for a^* and b^* , evaluates $\phi_{10} = (1 + w_1\beta_1/\delta_1)/b^*$ and $\phi_{20} = (1 + w_1\beta_1/\delta_1)/a^*$, and delivers $I(s, w_1) = (\phi_{10}s, \phi_{20}s)$. We thus see that the main step in the computation of $I_1(S, W_1)$

is simply a rerun of the procedure that produces I_0 replacing α_1 and $1 - \alpha$ by $\alpha_1 + \gamma_1$ and ϵ respectively. This makes the proposed interval $I_1(S, W_1)$ quite easy to use in practice.

Example 1. (Normal distribution) Let U, V, S_1, S_2 be independent random variables where $U \sim N_p(\mu_1, \sigma_1^2 I_p), V \sim N_q(\mu_2, \sigma_2^2 I_q), S_1/\sigma_1^2 \sim \chi_n^2$ and $S_2/\sigma_2^2 \sim \chi_m^2$. In this example the parameter of interest is $\sigma = \sigma_1^2/\sigma_2^2$. Accordingly, we have $T_1 = \|U\|^2$ and $T_2 = \|V\|^2$ with distributions $T_1/\sigma_1^2 \sim \chi_p^2(\|\mu_1\|^2/\sigma_1^2)$ and $T_2/\sigma_2^2 \sim \chi_q^2(\|\mu_2\|^2/\sigma_2^2)$ (so that $\lambda_1 = \lambda_2 = 0$). Taking $\mu_{10} = 0, \sigma_{10}^2 = 1$, Condition (A2) holds by the well known monotone likelihood ratio property of the noncentral chi-squared distribution. Also (A1), (A3), and (2.6) are satisfied by Theorem 2.3. Figures 1–3 give the coverage probability and percentage improvement in expected ratio of endpoints of $I_1(S, W_1)$ over the standard F -interval plotted against the noncentrality parameter $\lambda_1 = \|\mu_1\|^2/\sigma_1^2$ for selected values of n, m, p and nominal level $1 - \alpha = 0.95$. These numerical results indicate that for λ_1 close to zero and small values of n, m , the improvement in ratio of endpoints can be quite substantial, whereas the maximum coverage probability can reach 0.9549.

Example 2. (Exponential distribution) Let $X_1, \dots, X_n, n \geq 2$, and $Y_1, \dots, Y_m, m \geq 2$, be two independent random samples from exponential distributions $E(\mu_1, \sigma_1)$ and $E(\mu_2, \sigma_2)$ with densities $\frac{1}{\sigma_1} e^{-(x-\mu_1)/\sigma_1} I_{(\mu_1, \infty)}(x)$ and $\frac{1}{\sigma_2} e^{-(x-\mu_2)/\sigma_2} I_{(\mu_2, \infty)}(x)$ respectively. Here the parameter of interest is $\sigma = \sigma_1/\sigma_2$, and $S_1 = \sum_{i=1}^n (X_i - X_{(1)})$, $S_2 =$

$\sum_{i=1}^m (Y_i - Y_{(1)})$, $T_1 = X_{(1)} = \min\{X_i : i = 1, \dots, n\}$ and $T_2 = Y_{(1)} = \min\{Y_i : i = 1, \dots, m\}$ with distributions $S_1/\sigma_1 \sim \frac{1}{2}\chi_{2(n-1)}^2$, $S_2/\sigma_2 \sim \frac{1}{2}\chi_{2(m-1)}^2$, $T_1/\sigma_1 \sim E(\mu_1/\sigma_1, 1/n)$ and $T_2/\sigma_2 \sim E(\mu_2/\sigma_2, 1/m)$ (so that $\lambda_1 = \mu_1/\sigma_1$ and $\lambda_2 = \mu_2/\sigma_2$). Taking $\mu_{10} = 0, \sigma_{10} = 1$, Condition (A2) holds trivially whereas (A1), (A3), and (2.6) are satisfied by Theorem 2.3.

Example 3. (Inverse Gaussian distribution) Let $X_1, \dots, X_n, n \geq 2$, and $Y_1, \dots, Y_m, m \geq 2$, be two independent random samples from inverse Gaussian distributions with densities $\left(\frac{\sigma_i}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\sigma_i(x - \mu_i)^2}{2\mu_i^2 x}\right\}$, $x > 0, \mu_i > 0, \sigma_i > 0, i = 1, 2$. In this example the parameter μ_i is not a location parameter and consequently this model is not a purely location-scale model. Here the parameter of interest is $\sigma = \sigma_2/\sigma_1$ ($1/\sigma_i$ is often referred to as the dispersion parameter, so σ is the ratio of dispersion parameters). The standard confidence interval, I_0 , for σ is based on $S = S_1/S_2$, where $S_1 = \sum_{i=1}^n \left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right)$,

$S_2 = \sum_{i=1}^m \left(\frac{1}{Y_i} - \frac{1}{\bar{Y}}\right)$, with distributions $\sigma_1 S_1 \sim \chi_n^2$ and $\sigma_2 S_2 \sim \chi_m^2$. To improve on I_0 , we let $T_1 = n(\bar{X} - 1)^2/\bar{X}$ and $T_2 = m(\bar{Y} - 1)^2/\bar{Y}$ (so that $\lambda_1 = \lambda_2 = 0$), and note that when $\mu_i = 1$, $\sigma_i T_i$ has a χ_1^2 distribution, $i = 1, 2$. Taking $\mu_{10} = 1, \sigma_{10} = 1$, Condition (A2) is shown to hold in Kourouklis [12], whereas (A1), (A3), and (2.6) are satisfied by Theorem 2.3.

3 Improved intervals for ordered scale parameters

In this section we assume the model and the conditions of Section 2. Under the order restriction $\sigma_1 \leq \sigma_2$ we address the problem of constructing better intervals for σ_1 than the standard ones, i.e., those that do not take into account this restriction. With σ_{i0} , $i = 1, 2$, as in (A2), (A3), (A'2), and (A'3) we will need the following condition.

(C1) $\sigma_{10} = \sigma_{20}$.

The standard confidence interval for σ_1 with confidence coefficient $1 - \alpha$ and minimum ratio of endpoints is given by

$$I_1 = (S_1/b_1, S_1/a_1), \quad (3.1)$$

where a_1 and b_1 satisfy

$$\int_{a_1}^{b_1} g_1(x) dx = 1 - \alpha \quad \text{and} \quad a_1 g_1(a_1) = b_1 g_1(b_1). \quad (3.2)$$

We aim at constructing intervals of smaller ratio of endpoints that, in addition, maintain coverage probability above the nominal level $1 - \alpha$. In typical cases (such as Examples 1–3 in Section 2), the problem of interval estimation of σ_1 subject to the restriction $\sigma_1 \leq \sigma_2$ is invariant under the group of transformations $(T_1, S_1, T_2, S_2) \rightarrow (cT_1, cS_1, cT_2, cS_2)$, $c > 0$, and the class of equivariant intervals is $\mathcal{C} = \{(\psi_1^*(S_2/S_1, T_1/S_1, T_2/S_1)S_1, \psi_2^*(S_2/S_1, T_1/S_1, T_2/S_1)S_1) : \psi_1^* < \psi_2^* \text{ positive functions}\}$. For $V = S_2/S_1$ we first consider the subclass of intervals of the form $I = (\psi_1(V)S_1, \psi_2(V)S_1)$. For $v > 0$, we find $\psi_{10} = \psi_{10}(v)$ and $\psi_{20} = \psi_{20}(v)$ so that

$$\int_{1/\psi_{20}}^{1/\psi_{10}} x g_1(x) g_2(xv) dx = \int_{a_1}^{b_1} x g_1(x) g_2(xv) dx \quad \text{and} \quad \frac{1}{\psi_{10}^2} g_1\left(\frac{1}{\psi_{10}}\right) g_2\left(\frac{v}{\psi_{10}}\right) = \frac{1}{\psi_{20}^2} g_1\left(\frac{1}{\psi_{20}}\right) g_2\left(\frac{v}{\psi_{20}}\right). \quad (3.3)$$

As in (2.4), the existence and uniqueness of ψ_{10} and ψ_{20} follows from the continuity of $xg_1(x)g_2(xv)$ and the unimodality of $x^2g_1(x)g_2(xv)$ as functions of $x > 0$ upon using Lemma 4.1 and (A1). We then define a confidence interval for σ_1 by

$$I(S_1, V) = (\psi_1(V)S_1, \psi_2(V)S_1),$$

where

$$\psi_1(V) = \min\{\psi_{10}(V), 1/b_1\} \quad \text{and} \quad \psi_2(V) = \min\{\psi_{20}(V), 1/a_1\}. \quad (3.4)$$

Theorem 3.1 *Assume that (A1) holds. If $\sigma_1 \leq \sigma_2$ and*

$$P(b_1 g_2(b_1 V) > a_1 g_2(a_1 V)) > 0 \quad (3.5)$$

then we have the following.

- (i) $P(\psi_1(V) < 1/b_1) > 0$.
- (ii) $P(\sigma_1 \in I(S_1, V)) \geq P(\sigma_1 \in I_1) = 1 - \alpha$ and the inequality is strict unless $\sigma_1 = \sigma_2$.
- (iii) The ratio of endpoints of $I(S_1, V)$ is less than or equal to b_1/a_1 and is strictly less with positive probability.

Proof. (i) The second equality in (3.3), the unimodality of $x^2g_1(x)g_2(xv)$ as a function of x , and an argument analogous to that in the beginning of the proof of Theorem 2.2 yield $\{v > 0 : \psi_1(v) < 1/b_1\} = \{v > 0 : b_1g_2(b_1v) > a_1g_2(a_1v)\}$. Thus, the result follows from (3.5).

(ii), (iii) The conditional density of S_1/σ_1 given $V=v$ is $g_1(x|v; \sigma_1, \sigma_2) \propto xg_1(x)g_2(\frac{\sigma_1}{\sigma_2}xv)$ and thus for $\sigma_1 = \sigma_2$ it simplifies to $g_1(x|v) \propto xg_1(x)g_2(xv)$. Furthermore, (A1) and the relation $\sigma_1 \leq \sigma_2$ imply that $g_1(x|v; \sigma_1, \sigma_2)/g_1(x|v)$ is strictly increasing in $x > 0$ unless $\sigma_1 = \sigma_2$. The proof of the result is now as that of Theorem 2.2 replacing $W_1, a, b, f_1(x|w_1)$, and $f_1(x|w_1; \mu_1, \sigma_1)$ by $V, a_1, b_1, g_1(x|v)$, and $g_1(x|v; \sigma_1, \sigma_2)$ respectively.

For further improvement we study intervals in the larger class \mathcal{C} , which, in addition, employ $W_1 = T_1/S_1$ and $Z_1 = T_2/S_1$, i.e., $I = (\psi_1^*(V, W_1, Z_1)S_1, \psi_2^*(V, W_1, Z_1)S_1)$. We let $v > 0, w_1 > 0, z_1 > 0$ and determine $\psi_{10}^* = \psi_{10}^*(v, w_1, z_1)$, $\psi_{20}^* = \psi_{20}^*(v, w_1, z_1)$ so that

$$\int_{1/\psi_{20}^*}^{1/\psi_{10}^*} x^3 g_1(x)g_2(xv)h_1(xw_1)h_2(xz_1)dx = \int_{1/\psi_2}^{1/\psi_1} x^3 g_1(x)g_2(xv)h_1(xw_1)h_2(xz_1)dx \quad (3.6)$$

and

$$\frac{1}{\psi_{10}^{*4}} g_1\left(\frac{1}{\psi_{10}^*}\right) g_2\left(\frac{v}{\psi_{10}^*}\right) h_1\left(\frac{w_1}{\psi_{10}^*}\right) h_2\left(\frac{z_1}{\psi_{10}^*}\right) = \frac{1}{\psi_{20}^{*4}} g_1\left(\frac{1}{\psi_{20}^*}\right) g_2\left(\frac{v}{\psi_{20}^*}\right) h_1\left(\frac{w_1}{\psi_{20}^*}\right) h_2\left(\frac{z_1}{\psi_{20}^*}\right),$$

where $\psi_1 = \psi_1(v)$ and $\psi_2 = \psi_2(v)$ are as in (3.4). Setting

$$\psi_1^*(v, w_1, z_1) = \begin{cases} \min\{\psi_{10}^*(v, w_1, z_1), \psi_1(v)\} & , \text{ if } v > 0, w_1 > 0, z_1 > 0, \\ \psi_1(v) & , \text{ otherwise} \end{cases}$$

$$\psi_2^*(v, w_1, z_1) = \begin{cases} \min\{\psi_{20}^*(v, w_1, z_1), \psi_2(v)\} & , \text{ if } v > 0, w_1 > 0, z_1 > 0, \\ \psi_2(v) & , \text{ otherwise} \end{cases},$$

we obtain the interval

$$I(S_1, V, W_1, Z_1) = (\psi_1^*(V, W_1, Z_1)S_1, \psi_2^*(V, W_1, Z_1)S_1). \quad (3.7)$$

The next result establishes the domination of $I(S_1, V, W_1, Z_1)$ over $I(S_1, V)$ and hence over I_1 .

Theorem 3.2 *Assume that (A1), (A2), (A3), (A'2), (A'3), and (C1) hold. If $\sigma_1 \leq \sigma_2$ and*

$$P(W_1 > 0, Z_1 > 0, b_1g_2(b_1V) > a_1g_2(a_1V),$$

$$\frac{1}{\psi_1^2(V)} h_1\left(\frac{W_1}{\psi_1(V)}\right) h_2\left(\frac{Z_1}{\psi_1(V)}\right) > \frac{1}{\psi_2^2(V)} h_1\left(\frac{W_1}{\psi_2(V)}\right) h_2\left(\frac{Z_1}{\psi_2(V)}\right) > 0, \quad (3.8)$$

where $\psi_1(V)$ and $\psi_2(V)$ are as in (3.4) then we have the following.

(i) $P(\psi_{10}^*(V, W_1, Z_1) < \psi_1(V)) > 0$.

(ii) $P(\sigma_1 \in I(S_1, V, W_1, Z_1)) \geq P(\sigma_1 \in I(S_1, V)) \geq 1 - \alpha$.

(iii) The ratio of endpoints of $I(S_1, V, W_1, Z_1)$ is less than or equal to that of $I(S_1, V)$ and is strictly less with positive probability.

Proof. (i) Using (3.3), (3.4), the relations defining ψ_{10}^* , ψ_{20}^* , the unimodality of $x^4 g_1(x) g_2(xv) h_1(xw_1) h_2(xz_1)$, and the argument in the beginning of the proof of Theorem 2.2 we obtain that $\{v > 0, w_1 > 0, z_1 > 0 : \psi_{10}^*(v, w_1, z_1) < \psi_1(v)\} = \{v > 0, w_1 > 0, z_1 > 0 : b_1 g_2(b_1 v) > a_1 g_2(a_1 v),$

$$\frac{1}{\psi_1^2(v)} h_1\left(\frac{w_1}{\psi_1(v)}\right) h_2\left(\frac{z_1}{\psi_1(v)}\right) > \frac{1}{\psi_2^2(v)} h_1\left(\frac{w_1}{\psi_2(v)}\right) h_2\left(\frac{z_1}{\psi_2(v)}\right)\}.$$

Thus, the result follows from (3.8).

(ii), (iii) The conditional density of S_1/σ_1 given $V = v > 0$, $W_1 = w_1 > 0$, and $Z_1 = z_1 > 0$ is $g_1(x|v, w_1, z_1; \mu_i, \sigma_i) \propto$

$$x^3 g_1(x) g_2\left(\frac{\sigma_1}{\sigma_2} xv\right) h_1(xw_1; \mu_1, \sigma_1) h_2\left(\frac{\sigma_1}{\sigma_2} xz_1; \mu_2, \sigma_2\right) I_{(\max(0, \frac{\lambda_1}{w_1}, \frac{\lambda_2 \sigma_2}{z_1 \sigma_1}), \infty)}(x)$$

which for $\mu_i = \mu_{i0}$ and $\sigma_i = \sigma_{i0}$, $i = 1, 2$, simply becomes

$$g_1(x|v, w_1, z_1) \propto x^3 g_1(x) g_2(xv) h_1(xw_1) h_2(xz_1) I_{(0, \infty)}(x).$$

Thus (A1) to (C1) imply that $g_1(x|v, w_1, z_1; \mu_i, \sigma_i)/g_1(x|v, w_1, z_1)$ is nondecreasing in $x > 0$. Furthermore, when $b_1 g_2(b_1 v) > a_1 g_2(a_1 v)$ we have $\psi_1(v) = \psi_{10}(v)$ (see the proof of part i of Theorem 3.1). Hence, by (3.3), the last inequality in (3.8) can be written as

$\frac{1}{\psi_1} g_1\left(\frac{1}{\psi_1} |v, w_1, z_1\right) > \frac{1}{\psi_2} g_1\left(\frac{1}{\psi_2} |v, w_1, z_1\right)$. In other words, (3.8) is equivalent to

$P(W_1 > 0, Z_1 > 0, b_1 g_2(b_1 V) > a_1 g_2(a_1 V),$

$$\frac{1}{\psi_1(V)} g_1\left(\frac{1}{\psi_1(V)} |V, W_1, Z_1\right) > \frac{1}{\psi_2(V)} g_1\left(\frac{1}{\psi_2(V)} |V, W_1, Z_1\right) > 0.$$

The rest of the proof proceeds as in Theorem 2.2 replacing $a, b, f_1(x|w_1)$ and $f_1(x|w_1; \mu_1, \sigma_1)$ by $1/\psi_1, 1/\psi_2, g_1(x|v, w_1, z_1)$, and $g_1(x|v, w_1, z_1; \mu_i, \sigma_i)$ respectively.

To illustrate Conditions (3.5) and (3.8), we consider the case where the g_i 's and h_i 's are the gamma densities in Theorem 2.3. Using the relations defining a_1 and b_1 in (3.2), it is straightforward to show that (3.5) is equivalent to $P(V < \frac{\alpha_2 \beta_2}{\alpha_1 \beta_1}) > 0$ which clearly holds. Also, using (3.3), it can be seen that (3.8) is equivalent to

$$P(W_1 > 0, Z_1 > 0, V < \frac{\alpha_2 \beta_2}{\alpha_1 \beta_1}, \frac{W_1}{\delta_1} + \frac{Z_1}{\delta_2} < \frac{\gamma_1 + \gamma_2}{\alpha_1 + \alpha_2} \left(\frac{1}{\beta_1} + \frac{V}{\beta_2}\right)) > 0$$

which is satisfied. Thus, in particular, Conditions (3.5) and (3.8) hold for the normal, exponential, and inverse Gaussian models in Examples 1–3 of Section 2.

Remark 3.1. We comment on the computation of the improved intervals $I(S_1, V)$ and $I(S_1, V, W_1, Z_1)$ when the g_i 's and h_i 's are the gamma densities in Theorem 2.3. In this case, making an obvious scale transformation, the integrand in (3.3) becomes the gamma density with shape parameter $\alpha_1 + \alpha_2$ and scale parameter β_1 . Similarly, the integrand in (3.6) becomes the gamma density with shape parameter $\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2$ and scale β_1 . On the other hand, the computation of I_1 in (3.1) involves the gamma density

with shape parameter α_1 and scale β_1 . Therefore, the main steps for the computation of $I(S_1, V)$ and $I(S_1, V, W_1, Z_1)$ are, merely, reruns of the algorithm that produces the standard interval I_1 in (3.1). Thus, the computational difficulty of all three intervals is essentially the same.

Remark 3.2. It is clear from the proofs of Theorems 3.1 and 3.2 that the standard interval I_1 in (3.1) can be improved by choosing any subset of V, W_1, Z_1 . If, initially, one uses only W_1 the resulting improvement is also a standard interval in the sense that its construction does not exploit the restriction $\sigma_1 \leq \sigma_2$. Then better improvements can be obtained by employing in addition V and/or Z_1 .

We now present an application of the above results. Consider the balanced one way random effects linear model $Y_{ij} = \mu + \alpha_i + e_{ij}$, $i = 1, \dots, \kappa$, $j = 1, \dots, m$, where α_i 's and e_{ij} 's are independent random variables with distributions $\alpha_i \sim N(0, \sigma_0)$ and $e_{ij} \sim N(0, \sigma_1)$. The minimal sufficient statistic (S_1, S_2, \bar{Y}) has independent components $S_1 = \sum_{i=1}^{\kappa} \sum_{j=1}^m (Y_{ij} - \bar{Y}_i)^2 \sim \sigma_1 \chi_n^2$, $S_2 = m \sum_{i=1}^{\kappa} (\bar{Y}_i - \bar{Y})^2 \sim (\sigma_1 + m\sigma_0) \chi_{\kappa-1}^2$, $\bar{Y} = \sum_{i=1}^{\kappa} \sum_{j=1}^m Y_{ij} / (m\kappa) \sim N\left(\mu, \frac{\sigma_1 + m\sigma_0}{m\kappa}\right)$, where $\bar{Y}_i = \sum_{j=1}^m Y_{ij} / m$ and $n = \kappa(m-1)$. The standard interval for σ_1 is based on S_1 whereas better ones are given in Kubokawa et al. [15]. The latter improve on coverage probability while maintaining the same ratio of endpoints as the standard interval. Setting $\sigma_2 = \sigma_1 + m\sigma_0$ and $T_2 = m\kappa \bar{Y}^2 \sim \sigma_2 \chi_1^2 \left(\frac{m\kappa \mu^2}{\sigma_2}\right)$, Theorems 3.1 and 3.2 can be used to provide improvements in terms of both coverage probability and ratio of endpoints by employing S_2/S_1 and/or $m\kappa \bar{Y}^2/S_1$. The magnitude of the improvement of $I_1(S_1, V)$ over I_1 (for the above model) was investigated numerically for selected values of κ, m and nominal level $1 - \alpha = 0.95$. In Figures 4 and 5, the coverage probability and percentage improvement in expected ratio of endpoints are plotted against σ_0 . (Note that these quantities are functions of $\sigma_2/\sigma_1 = 1 + m\sigma_0/\sigma_1$, so σ_1 was taken one.) The numerical results indicate that $I_1(S_1, V)$ performs better, in terms of maximum improvement, when the number of levels of the random effect is large relative to the cell size.

Theorems 3.1 and 3.2 can be extended towards two directions: one is to treat more than two populations and the other one is to allow T_j to have a scale parameter β_j rather than σ_j . Specifically, consider the model where one observes independent random variables S_1, \dots, S_κ and T_1, \dots, T_m such that $S_i/\sigma_i \sim g_i(x)I_{(0,\infty)}(x)$ and $T_j/\beta_j \sim h_j(x; \mu_j, \beta_j)I_{(\lambda_j, \infty)}(x)$. Then under the restrictions $\sigma_1 \leq \sigma_i$, $i = 2, \dots, \kappa$, $\sigma_1 \leq \beta_j$, $j = 1, \dots, m$ and some obvious modifications of (A1) to (C1), one can derive improved intervals for σ_1 which are extensions of those in Theorems 3.1 and 3.2. An interesting situation where the above applies is the problem of interval estimation of the error variance in a general balanced mixed linear model. This model is studied in Mathew and Sinha [21] and Mathew et al. [22], where the latter presents improved point estimators of the error variance, σ_1 say. Improved intervals for σ_1 , however, are

not available in the literature. The model can be described by means of independent canonical variables $S_i \sim \sigma_i \chi_{q_i}^2$ and $T_j \sim \beta_j \chi_{p_j}^2(\delta_j)$ where $\sigma_1 \leq \sigma_i$ and $\sigma_1 \leq \beta_j$ for $i = 2, \dots, \kappa$ and $j = 1, \dots, m$. (For instance, S_1 is the error sum of squares.) We refer to the above articles for a detailed account on this canonical form of the model. Thus, the interval estimation of σ_1 falls into the above framework and improved procedures can be produced.

We close this section by noting that the standard minimum ratio of endpoints interval for σ_2 , which is of the form $(S_2/b_2, S_2/a_2)$, can be improved (under $\sigma_1 \leq \sigma_2$) by an interval of the form $(\phi_1(S)S_2, \phi_2(S)S_2)$. The construction of the latter is analogous to that in Theorem 3.1. Details of this construction are given in Iliopoulos and Kourouklis [8].

4 Appendix

Lemma 4.1 *Let $g(x)$ be a density function on $(0, \infty)$ with $g(x) > 0$ for every $x > 0$ and such that*

$$g(c_1x)/g(c_2x) \text{ is strictly increasing in } x \text{ for } 0 < c_1 < c_2. \quad (4.1)$$

Then $g(x)$ is continuous on $(0, \infty)$, $xg(x)$ is unimodal, and $\lim_{x \rightarrow 0} xg(x) = \lim_{x \rightarrow \infty} xg(x) = 0$.

Proof. Condition (4.1) means that the scale family $\{c^{-1}g(x/c) : c > 0\}$ has the monotone likelihood ratio property and hence that $-\log(e^y g(e^y))$ is a convex function of $y \in (-\infty, \infty)$, cf. Lehmann [18], p.510. Consequently, $g(x)$ is continuous on $(0, \infty)$. In addition, $f(y) = e^y g(e^y)$ is either monotone or increasing and then decreasing. The former obviously cannot hold since $f(y)$ is a strictly positive density on $(-\infty, \infty)$. Therefore, for $x > 0$, $xg(x) = f(\log x)$ is increasing and then decreasing function of x . In addition, $xg(x)$ cannot be constant on any interval (x_1, x_2) , since if it were then setting $x_0 = (x_1 x_2)^{1/2}$ and $c_2 = x_0/x_1$ we would have $g(x)/g(c_2x) = c_2$ for all $x \in (x_1, x_0)$ which contradicts (4.1). Now for small x , $xg(x) < \int_x^{2x} g(v)dv$, implying $\lim_{x \rightarrow 0} xg(x) = 0$. Similarly for large x , $xg(x) < 2 \int_{x/2}^x g(v)dv$, and thus $\lim_{x \rightarrow \infty} xg(x) = 0$.

Lemma 4.2 *Suppose that $b_1(x)$ and $b_2(x)$ are densities supported on the same interval of the real line and $b_2(x)/b_1(x)$ is strictly increasing. If X is a random variable with density $b_1(x)$ or $b_2(x)$ and $h(x)$ is nondecreasing but not constant on the common support of $b_1(x)$ and $b_2(x)$, then $E_{b_2}h(X) > E_{b_1}h(X)$ (provided expectations exist).*

Proof. It is given in Lehmann [18], p.85.

Lemma 4.3 *Let X_i and Y be independent random variables, where X_i has density $f_i(x)$, $i = 1, 2$, Y has density $f(x)$, X_1, Y are supported on $(0, \infty)$ and X_2 is supported on (μ, ∞) , $\mu \geq 0$. Furthermore suppose that $f_2(x)/f_1(x)$ is nondecreasing on $(0, \infty)$ and $f(c_1x)/f(c_2x)$ is strictly increasing for $0 < c_1 < c_2$. If $h_i(x)$ is the density of X_i/Y , $i = 1, 2$, then $h_2(x)/h_1(x)$ is strictly increasing in $x > 0$ unless $f_1 = f_2$.*

Proof. It can be given using arguments of total positivity (Karlin [11], composition theorem) or variation diminishing transformations (Brown et al. [1]). See also Iliopoulos and Kourouklis [10] for a direct proof.

Lemma 4.4 *Let $Y_i, i = 1, 2$, be independent random variables with densities $g_i(x)$, respectively, supported on $(0, \infty)$. Suppose that $g_i(c_1x)/g_i(c_2x)$ is strictly increasing in $x > 0$ for $0 < c_1 < c_2$ and denote by $g(x)$ the density of Y_1/Y_2 . Then $g(c_1x)/g(c_2x)$ is also strictly increasing in $x > 0$ for $0 < c_1 < c_2$.*

Proof. It is an application of Lemma 4.3 with $X_1 = Y_1/c_2$, $X_2 = Y_1/c_1$, and $Y = Y_2$.

Lemma 4.5 *Let $f(x)$ be a positive valued function on $(0, \infty)$ such that $xf(x)$ is unimodal and $cf(c) = df(d)$ for $0 < c < d$. Suppose also that for $0 < a < b$, neither of the intervals (a, b) and (c, d) is a proper subset of the other. Then the relation $bf(b) > (\text{resp. } <) af(a)$ implies $d > (\text{resp. } <) b$.*

Proof. The argument is along the lines of that in the proof of Lemma A.2 of Goutis and Casella [5].

Lemma 4.6 *Suppose that $f_1(x)$ is an integrable function on $(0, \infty)$ such that $f_1(x) > 0$ for every $x > 0$ and, for $\mu \geq 0$, $f_2(x)$ is an integrable function on (μ, ∞) . Furthermore assume that for constants $a < b$ and $c < d$ in the interval (μ, ∞) we have*

$$\int_c^d f_1(x)dx = \int_a^b f_1(x)dx . \quad (4.2)$$

If $d > b$ and $f_2(x)/f_1(x)$ is strictly increasing (resp. nondecreasing) in (μ, ∞) or if $d < b$ and $f_2(x)/f_1(x)$ is strictly decreasing (resp. nonincreasing) in (μ, ∞) , then

$$\int_c^d f_2(x)dx > (\text{resp. } \geq) \int_a^b f_2(x)dx . \quad (4.3)$$

Proof. We prove the result only when $d > b$ and $f_2(x)/f_1(x)$ is strictly increasing. Then, by (4.2), the possible cases are:

- (i) $a < c < b < d$,
- (ii) $a < b \leq c < d$.

Assume that (i) holds. Then $\int_c^d f_2(x)dx - \int_a^b f_2(x)dx = \int_b^d f_2(x)dx - \int_a^c f_2(x)dx > \frac{f_2(b)}{f_1(b)} \int_b^d f_1(x)dx - \frac{f_2(c)}{f_1(c)} \int_a^c f_1(x)dx = \left(\frac{f_2(b)}{f_1(b)} - \frac{f_2(c)}{f_1(c)} \right) \int_a^c f_1(x)dx > 0$. The proof under (ii) is similar.

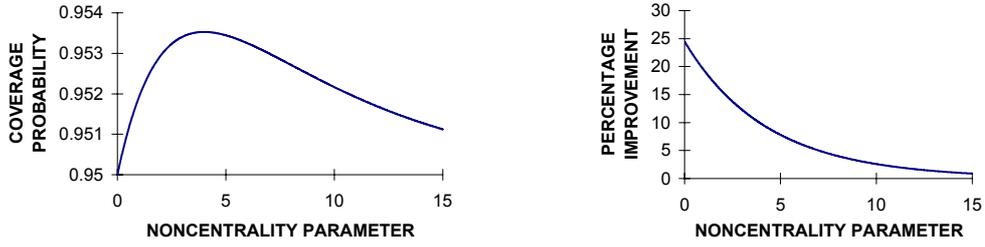


Fig.1. $n=m=3, p=3$. Maximum coverage probability 0.9535. Maximum percentage improvement in expected ratio of endpoints 24.5%.

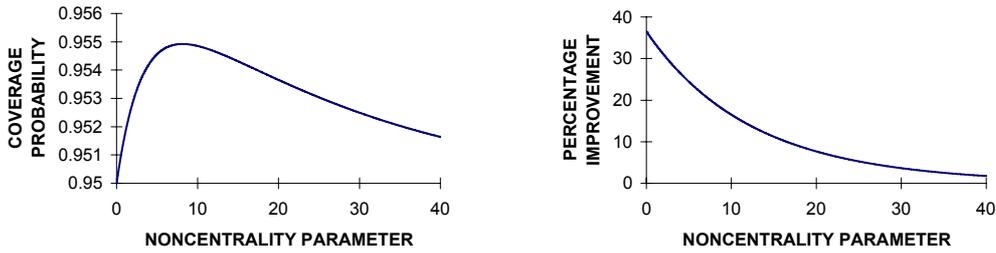


Fig.2. $n=m=3, p=10$. Maximum coverage probability 0.9549. Maximum percentage improvement in expected ratio of endpoints 36.5%.

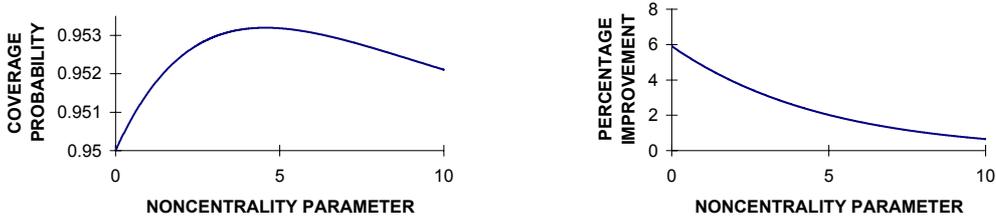


Fig.3. $n=20, m=30, p=10$. Maximum coverage probability 0.9532. Maximum percentage improvement in expected ratio of endpoints 5.91%.

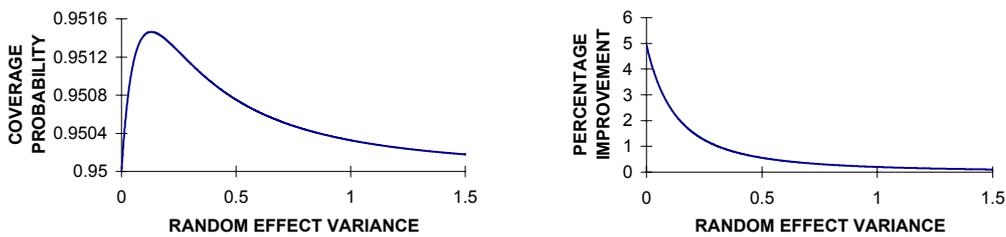


Fig.4. $\kappa=5, m=6, n=\kappa(m-1)=25$. Maximum coverage probability 0.9515. Maximum percentage improvement in expected ratio of endpoints 4.93%.

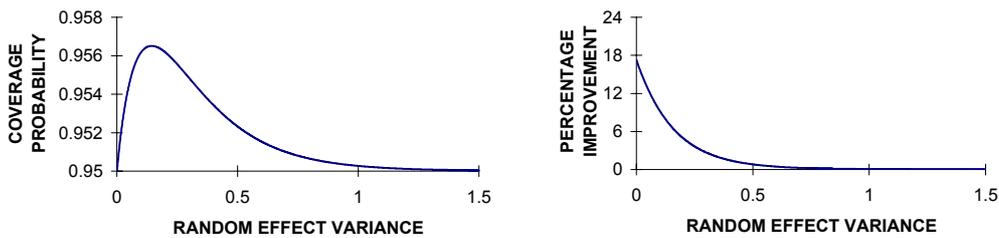


Fig.5. $\kappa=25, m=2, n=\kappa(m-1)=25$. Maximum coverage probability 0.9565. Maximum percentage improvement in expected ratio of endpoints 17.25%.

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