Conditional independence of blocked ordered data

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Abstract

In this paper, we prove that blocks of ordered data formed by some conditioning events are mutually independent. We establish this result by considering the usual order statistics, progressively censored order statistics, and concomitants of order statistics.

Keywords and phrases: Conditional independence; order statistics; progressively censored order statistics; concomitants of order statistics.

1 Introduction

Let $X_1, \cdots, X_n$ be independent and identically distributed (iid) random variables from some distribution with cumulative distribution function (cdf) F, and $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ be the corresponding order statistics (OS). Properties related to conditional independence of the ordered observations are mainly based on the Markov property. It is well-known [see, for example, Arnold, Balakrishnan and Nagaraja (2008) and David and Nagaraja (2003)] that when F is continuous, the ordered sample forms a Markov chain and so, conditional on $X_{j:n} = x$, the random vectors $(X_{1:n}, \cdots, X_{j-1:n})$ and $(X_{j+1:n}, \cdots, X_{n:n})$ are independent. On the other hand, if F is discrete with support containing more than two points, the order statistics do not form a Markov chain due to the possibility of ties [see Nagaraja (1982)]. However, they can become Markovian by conditioning on some suitable events; see, for example, Rüshendorf (1982) and Nagaraja (1986).

This situation is similar in the case of ordered data arising from some censoring schemes as well. Consider, for example, progressive Type-II right censoring wherein at the times of observed failures a pre-fixed number of surviving units are withdrawn. If the underlying distribution is continuous, these progressively censored order statistics also form a Markov chain; see Balakrishnan and Aggarwala (2000). This is not true in the discrete case as shown by Balakrishnan and Dembińska (2008), but here again conditioning on some suitable events may lead to Markovian structure.

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While deiving the exact distribution of the maximum likelihood estimator (MLE) of the exponential mean parameter under different forms of censoring and its stochastic monotonicity, Balakrishnan and Iliopoulos (2008) required the distribution of a function of exponential OS that could be expressed as a discrete mixture with the value of the mixing variable D (say) being precisely the number of the OS that are at most some pre-fixed number T. It was observed by these authors, in this case, that conditional on $D = d$, the first d OS (i.e., those that are at most T) are independent of the rest (i.e., those that are larger than T). This property results from the factorization of the joint probability density function (pdf) of OS. It turns out that this property holds in more general settings wherein the underlying distribution may be either continuous or discrete, and the cut-points being more than one. Furthermore, this conditional block independence property holds also for progressively Type-I and Type-II right censored OS as well as for concomitants of OS and generalized OS.

The rest of this paper proceeds as follows. In Section 2, the block independence result is presented for the case of (usual) OS. In Section 3, we extend this result for progressively Type-I and Type-II right censored OS (PCOS). In Section 4, we show the conditional independence of concomitants of blocked OS. Finally, some concluding remarks are made in Section 5.

2 Usual order statistics

Let $X_1, \cdots, X_n$ be independent random variables from some distribution with cdf F and $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ be the corresponding order statistics. For some fixed $T \in \mathbb{R}$, let $D = \#\{X's \leqslant T\}$. Then, the following theorem presents the conditional independence result for the usual order statistics.

Theorem 1. *Conditional on $D = d$, the vectors $(X_{1:n}, \cdots, X_{d:n})$ and $(X_{d+1:n}, \cdots, X_{n:n})$ are mutually independent with*

$$\begin{aligned} (X_{1:n}, \cdots, X_{d:n}) &\stackrel{\mathrm{d}}{=} (V_{1:d}, \cdots, V_{d:d}), \\ (X_{d+1:n}, \cdots, X_{n:n}) &\stackrel{\mathrm{d}}{=} (W_{1:n-d}, \cdots, W_{n-d:n-d}), \end{aligned} \tag{1}$$

where $V_1, \cdots, V_d$ are iid from F but right-truncated at T, and $W_1, \cdots, W_{n-d}$ are iid from F but left-truncated at T.

Proof. Recall that D has the binomial distribution $\mathcal{B}(n, F(T))$, irrespective of whether F is discrete or continuous, with probability mass function (pmf)

$$\mathsf{P}(D = d) = \binom{n}{d} \{F(T)\}^d \{1 - F(T)\}^{n-d}, \quad d = 0, 1, \cdots, n. \tag{2}$$

When $d = 0$ or n, one of the two random vectors has zero dimension and the result holds in this case trivially. So, in what follows, we assume $d \in \{1, \cdots, n-1\}$.

Let us first consider the case when F is an absolutely continuous distribution with f as the corresponding pdf. Then, the conditional joint pdf of the order statistics is given by

$$g(x_1, \cdots, x_n | D = d) = \frac{n! \prod_{i=1}^n f(x_i)}{\mathsf{P}(D = d)} \, I(x_1 < \cdots < x_d \leqslant T < x_{d+1} < \cdots < x_n),$$

where $I(A)$ denotes the indicator function for event A. With (2), we then have

$$\begin{aligned} g(x_1, \cdots, x_n | D = d) &= \frac{n! \prod_{i=1}^n f(x_i)}{\frac{n!}{d!(n-d)!}\{F(T)\}^d \{1 - F(T)\}^{n-d}} \\ &\qquad \times I(x_1 < \cdots < x_d \leqslant T < x_{d+1} < \cdots < x_n) \\ &= \left\{ d! \prod_{i=1}^{d} \frac{f(x_i)}{F(T)} \, I(x_1 < \cdots < x_d \leqslant T) \right\} \\ &\qquad \times \left\{ (n-d)! \prod_{i=d+1}^{n} \frac{f(x_i)}{1 - F(T)} I(T < x_{d+1} < \cdots < x_n) \right\} \end{aligned}$$

which proves the result in (1).

In the case when F is discrete, let us consider the integral representation of the joint pmf of the order statistics given by

$$g(x_1, \cdots, x_n) = n! \int_{F(x_1-)}^{F(x_1)} \cdots \int_{F(x_d-)}^{F(x_d)} \int_{F(x_{d+1}-)}^{F(x_{d+1})} \cdots \int_{F(x_n-)}^{F(x_n)} \mathrm{d}u_n \cdots \mathrm{d}u_{d+1} \mathrm{d}u_d \cdots \mathrm{d}u_1,$$

for $x_1 \leqslant \cdots \leqslant x_n$; see Arnold, Balakrishnan and Nagaraja (2008) and David and Nagaraja (2003). Clearly, the conditional joint pmf of the order statistics, given $D = d$, is then

$$g(x_1, \cdots, x_n | D = d) = \frac{g(x_1, \cdots, x_n)}{\binom{n}{d}\{F(T)\}^d \{1 - F(T)\}^{n-d}},$$

for $x_1 \leqslant \cdots \leqslant x_d \leqslant T < x_{d+1} \leqslant \cdots \leqslant x_n$, which in turn yields

$$\begin{aligned} g(x_1, \cdots, x_n | D = d) &= \frac{d!}{\{F(T)\}^d} \int_{F(x_1-)}^{F(x_1)} \cdots \int_{F(x_d-)}^{F(x_d)} \mathrm{d}u_d \cdots \mathrm{d}u_1 \\ &\qquad \times \frac{(n-d)!}{\{1 - F(T)\}^{n-d}} \int_{F(x_{d+1}-)}^{F(x_{d+1})} \cdots \int_{F(x_n-)}^{F(x_n)} \mathrm{d}u_n \cdots \mathrm{d}u_{d+1}. \end{aligned}$$

Now, let $F_1(x) = \frac{F(x)}{F(T)}$ and $F_2(x) = \frac{F(x) - F(T)}{1 - F(T)}$ denote, respectively, the cdf's of the right- and left-truncated versions of F at T. Then, by making the transformations $v_i = \frac{u_i}{F(T)}$ for $i = 1, \cdots, d$ and $w_i = \frac{u_i - F(T)}{1 - F(T)}$ for $i = d + 1, \cdots, n$, we readily have

$$g(x_1, \cdots, x_n | D = d) = d! \int_{F_1(x_1-)}^{F_1(x_1)} \cdots \int_{F_1(x_d-)}^{F_1(x_d)} \mathrm{d}v_d \cdots \mathrm{d}v_1$$

$$\times(n-d)! \int_{F_2(x_{d+1}-)}^{F_2(x_{d+1})} \cdots \int_{F_2(x_n-)}^{F_2(x_n)} \mathrm{d}w_n \cdots \mathrm{d}w_{d+1},$$

which establishes the required result. $\square$

Remark 1. In the case of absolutely continuous distributions, the result can also be viewed from the Markovian property of order statistics. Indeed, the conditional distribution of $(X_{1:n}, \cdots, X_{d:n}, X_{d+1:n}, \cdots, X_{n:n})$, given $D = d$, is exactly the same as the conditional distribution of $(X_{1:n+1}, \cdots, X_{d:n+1}, X_{d+2:n+1}, \cdots, X_{n+1:n+1})$, given $X_{d+1:n+1} = T$. In this case, $(X_{1:n+1}, \cdots, X_{d:n+1})$ and $(X_{d+2:n+1}, \cdots, X_{n+1:n+1})$ become independent and are distributed as OS from the above stated right- and left-truncated distributions respectively.

Along the same lines, the following theorem states the conditional independence in the case of multiple cut-points. Let $-\infty \equiv T_0 < T_1 < \cdots < T_p < \infty$, and let $D_j = \#\{X's \in (T_{j-1}, T_j]\}$, $j = 1, \cdots, p$. For simplicity in notation, for any vector $(d_1, \cdots, d_p)$, we set $d_0 \equiv 0$ and $d_{(j)} = \sum_{i=0}^{j} d_i$.

Theorem 2. *Conditional on $(D_1, \cdots, D_p) = (d_1, \cdots, d_p)$, the random vectors*

$$(X_{1:n}, \cdots, X_{d_1:n}), \quad (X_{d_1+1:n}, \cdots, X_{d_1+d_2:n}), \quad \cdots \quad , \quad (X_{d_{(p)}+1:n}, \cdots, X_{n:n})$$

are mutually independent with

$$\begin{aligned} \left(X_{d_{(j-1)}+1:n}, \cdots, X_{d_{(j)}:n}\right) &\stackrel{\mathrm{d}}{=} \left(V_{1:d_j}^{(j)}, \cdots, V_{d_j:d_j}^{(j)}\right) \quad \textit{for } j = 1, \cdots, p, \\ \left(X_{d_{(p)}+1:n}, \cdots, X_{n:n}\right) &\stackrel{\mathrm{d}}{=} \left(V_{1:n-d_{(p)}}^{(p+1)}, \cdots, V_{n-d_{(p)}:n-d_{(p)}}^{(p+1)}\right), \end{aligned} \tag{3}$$

where $V_1^{(j)}, \cdots, V_{d_j}^{(j)}$, $j = 1, \cdots, p$, are iid from F but doubly truncated in the interval $(T_{j-1}, T_j]$, and $V_1^{(p+1)}, \cdots, V_{n-d_{(p)}}^{(p+1)}$ are iid from F but left-truncated at T_p.

Proof. The result in (3) follows by induction along the same lines as in the proof of Theorem 1. $\square$

3 Progressively right censored order statistics

3.1 Type-I progressive censoring

Let $X_1, \cdots, X_n$ be a random sample from some lifetime distribution with cdf F. In Type-I progressive right censoring, m timepoints $T_1 < \cdots < T_m$ are pre-fixed, and R_i surviving units are censored at time T_i, $i = 1, \cdots, m-1$. The experiment terminates at time T_m, at which point all the remaining R_m units are censored. The observed failures times form the so-called Type-I progressively right censored order statistics; see, for example, Balakrishnan and Aggarwala (2000) and Balakrishnan (2007). Clearly, there is a positive probability that at some time T_i, $i < m$, less than R_i units are surviving in which case the experiment would simply terminate at T_i instead of at T_m. Although this set-up has been explained from a life-testing point of view which usually involves only non-negative random variables, these Type-I progressively right censored order statistics can clearly be defined for discrete and/or non-positive random variables as well.

As before, let us set $T_0 = -\infty$ and $D_j = \#\{X's \in (T_{j-1}, T_j]\}$. Let $(Y_{D_{(j-1)}+1}, \cdots, Y_{D_{(j)}})$ be the vector consisting of the ordered X's falling in the interval $(T_{j-1}, T_j]$. (Of course, in case when $D_j = D_{(j)} - D_{(j-1)} = 0$ for some j, then the corresponding Y-vector is of zero dimension.) Note that the random variables $Y_1, \cdots, Y_{D_{(m)}}$ are the Type-I progressively right censored order statistics (Type-I PCOS). Then, the following theorem establishes the conditional block independence of Type-I PCOS.

Theorem 3. *Conditional on $(D_1, \cdots, D_m) = (d_1, \cdots, d_m)$, the random vectors*

$$(Y_1, \cdots, Y_{d_1}), \quad (Y_{d_1+1}, \cdots, Y_{d_{(2)}}), \quad \cdots \quad , \quad (Y_{d_{(m-1)}+1}, \cdots, Y_{d_{(m)}})$$

are mutually independent with

$$\big(Y_{d_{(j-1)}+1}, \cdots, Y_{d_{(j)}}\big) \stackrel{\mathrm{d}}{=} \big(V^{(j)}_{1:d_j}, \cdots, V^{(j)}_{d_j:d_j}\big), \tag{4}$$

where $V_1^{(j)}, \cdots, V_{d_j}^{(j)}$ are iid from F but douly truncated in the interval $(T_{j-1}, T_j]$, for $j = 1, \cdots, m$.

Proof. In what follows, we use the convention $\binom{0}{j} = 1$ when $j = 0$ and 0 otherwise. Then, the joint pdf of $D_1, \cdots, D_m, Y_1, \cdots, Y_{D_{(m)}}$ is

$$\begin{aligned} g(d_1, \ldots, d_m, y_1, \ldots, y_{d_{(m)}}) &= \prod_{i=1}^{m} \binom{\max\{n - d_{(i-1)} - R_{(i-1)}, 0\}}{d_i} \\ &\times d_i! \Big\{ \prod_{j=1}^{d_i} f(y_{d_{(i-1)}+j}) \Big\} \{1 - F(T_i)\}^{\zeta_i} \ I(T_{i-1} < y_{d_{(i-1)}+1} \leqslant \cdots \leqslant y_{d_{(i)}} \leqslant T_i), \end{aligned} \tag{5}$$

for $(d_1, \cdots, d_m) \in \mathcal{D} = \{(d_1, \cdots, d_m) \in \mathbb{Z}_+^m;\ 0 \leqslant d_i \leqslant \max(n - d_{(i-1)} - R_{(i-1)}, 0),\ i = 1, \cdots, m\}$, where $\zeta_i = \min\{R_i, \max(n - d_{(i)} - R_{(i-1)}, 0)\}$, $i = 1, \cdots, m$. Moreover, the joint pmf of $D_1, \cdots, D_m$ is

$$g(d_1, \cdots, d_m) = \prod_{i=1}^{m} \binom{\max\{n - d_{(i-1)} - R_{(i-1)}, 0\}}{d_i} \{F(T_i) - F(T_{i-1})\}^{d_i} \{1 - F(T_i)\}^{\zeta_i}$$

for $(d_1, \cdots, d_m) \in \mathcal{D}$. Hence, for any $(d_1, \cdots, d_m) \in \mathcal{D}$, we obtain

$$\begin{aligned} g(y_1, \cdots, y_{d_{(m)}} | d_1, \cdots, d_m) &= \frac{g(d_1, \cdots, d_m, y_1, \cdots, y_{d_{(m)}})}{g(d_1, \cdots, d_m)} \\ &= \prod_{i=1}^{m} d_i! \Big\{ \prod_{j=1}^{d_i} \frac{f(y_{d_{(i-1)}+j})}{F(T_i) - F(T_{i-1})} \Big\} \ I\big(T_{i-1} < y_{d_{(i-1)}+1} \leqslant \cdots \leqslant y_{d_{(i)}} \leqslant T_i\big), \end{aligned}$$

which proves the assertion for the continuous case.

Next, for proving the assertion for discrete F, we can use the integral representation of the pmf based on the probability integral transformation. More specifically, the joint pmf of $D_1, \cdots, D_m, Y_1, \cdots, Y_{D_{(m)}}$ has the same form as in (5), but with the d_i-fold integral $\int_{F(y_{d_{(i-1)}+1}-)}^{F(y_{d_{(i-1)}+1})} \cdots \int_{F(y_{d_{(i)}}-)}^{F(y_{d_{(i)}})} \mathrm{d}u_{d_i} \cdots \mathrm{d}u_1$ in place of $\prod_{j=1}^{d_i} f(y_{d_{(i-1)}+j})$. Then, upon dividing it by the joint pmf of $D_1, \cdots, D_m$ which has the same form as before and making the change of variables $v_j = \frac{u_j - F(T_{i-1})}{F(T_i) - F(T_{i-1})}$, $j = 1, \cdots, d_i$, for the u's in the i-th interval, we arrive at the required result. □

Remark 2. Note that by choosing $R_1 = \cdots = R_{m-1} = 0$, Theorem 3 reduces to Theorem 2. Hence, the above proof applies to that theorem as well, although a direct proof for Theorem 2 along the lines of Theore 1 would be much simpler.

Remark 3. The above result has been used recently by Balakrishnan, Han and Iliopoulos (2008) to establish that the sum of Type-I PCOS under exponentiality is stochastically increasing with respect to the mean θ of the underlying exponential distribution, a property that is essential to construct exact confidence intervals for the parameter θ.

3.2 Type-II progressive censoring

Let $(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)})$ be the Type-II progressively right censored order statistics (Type-II PCOS) with progressive censoring scheme $(R_1, \cdots, R_m)$ from some distribution with cdf $F(\cdot)$. Under this scheme, n iid units are placed under a life-test, and when the first failure occurs, R_1 of the $n-1$ surviving units are randomly withdrawn; at the time of the next failure, R_2 of the $(n-2-R_1)$ surviving units are randomly withdrawn, and so on; finally, at the time of the m-th failure, all the surviving R_m units are withdrawn. The m failure times that are observed in this manner, denoted by $X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)}$, are referred to as Type-II progressively right censored order statistics (Type-II PCOS). For a fixed point T, let D denote the number of Type-II PCOS that do not exceed T. We then have the following conditional independence result for these Type-II PCOS.

Theorem 4. *Conditional on $D = d$, the random vectors $(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d:m:n}^{(R_1,\cdots,R_m)})$ and $(X_{d+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)})$ are mutually independent with*

$$\begin{aligned} \big(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d:m:n}^{(R_1,\cdots,R_m)}\big) &\overset{\mathrm{d}}{=} \big(V_{1:d:d+K_{(d)}}^{(K_1,\cdots,K_d)}, \cdots, V_{d:d:d+K_{(d)}}^{(K_1,\cdots,K_d)}\big), \\ \big(X_{d+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)}\big) &\overset{\mathrm{d}}{=} \big(W_{1:m-d:n-d-R_{(d)}}^{(R_{d+1},\cdots,R_m)}, \cdots, W_{m-d:m-d:n-d-R_{(d)}}^{(R_{d+1},\cdots,R_m)}\big), \quad (6) \end{aligned}$$

where $V_1, \cdots, V_{d+K_{(d)}}$ are iid from F but right-truncated at T, $W_1, \cdots, W_{n-d-R_{(d)}}$ are iid from F but left-truncated at T, $R_{(d)} = \sum_{i=1}^{d} R_i$, $K_{(d)} = \sum_{i=1}^{d} K_i$, and $(K_1, \cdots, K_d)$ have joint pmf

$$p(k_1, \cdots, k_d) = \left[\sum_{k=0}^{d} \frac{(-1)^k \{1 - F(T)\}^{\sum_{i=d-k+1}^{d}(1+R_i)}}{\{\prod_{j=1}^{d-k} \sum_{i=j}^{d-k}(1+R_i)\}\{\prod_{j=d-k+1}^{d} \sum_{i=d-k+1}^{j}(1+R_i)\}} \right]^{-1}$$

$$\times \prod_{i=1}^{d} \frac{1}{\sum_{j=i}^{d}(1+k_j)} \binom{R_i}{k_i} F(T)^{k_i+1} \{1 - F(T)\}^{R_i - k_i}$$

for $0 \leqslant k_i \leqslant R_i$, $i = 1, \cdots, d$.

Proof. First, let us consider the case when F is a continuous distribution. Recall that the joint pdf of the PCOS is

$$f(x_1, \cdots, x_m) = \left\{ \prod_{i=1}^{m} \sum_{j=i}^{m} (1 + R_j) \right\} \prod_{i=1}^{m} f(x_i) \{1 - F(x_i)\}^{R_i} \; I(x_1 < \cdots < x_m).$$

Conditional on $D = d$, the joint pdf remains proportional to the above quantity, and so the independence result follows immediately since the joint pdf factors into two terms corresponding to the two blocks.

For deriving the conditional distributions, conditional on $D = d$, we start with the pmf of D given by [see Xie, Balakrishnan and Han (2008)]

$$\mathsf{P}(D = d) = \sum_{k=0}^{d} \frac{(-1)^k \{1 - F(T)\}^{\sum_{i=d-k+1}^{m}(1+R_i)} \prod_{j=1}^{d} \sum_{i=j}^{m}(1+R_i)}{\{\prod_{j=1}^{d-k} \sum_{i=j}^{d-k}(1+R_i)\}\{\prod_{j=d-k+1}^{d} \sum_{i=d-k+1}^{j}(1+R_i)\}}, \quad d = 0, 1, \cdots, m. \tag{7}$$

The conditional pdf of PCOS, given $D = d$, is clearly

$$f(x_1, \cdots, x_m | D = d) = \frac{f(x_1, \ldots, x_m)}{\mathsf{P}(D = d)} I(x_1 < \cdots \leqslant T < x_{d+1} < \cdots < x_m).$$

Observe now that for $x_1 < \cdots < x_d \leqslant T < x_{d+1} < \cdots < x_m$, $f(x_1, \cdots, x_m)$ equals

$$\left\{ \prod_{i=1}^{d} \sum_{j=i}^{m} (1 + R_j) \prod_{i=1}^{d} f(x_i) \{1 - F(x_i)\}^{R_i} \right\} \left\{ \prod_{i=d+1}^{m} \sum_{j=i}^{m} (1 + R_j) \prod_{i=d+1}^{m} f(x_i) \{1 - F(x_i)\}^{R_i} \right\}$$

$$= \left\{ \prod_{i=1}^{d} \sum_{j=i}^{m} (1 + R_j) \prod_{i=1}^{d} f(x_i) \sum_{k_i=0}^{R_i} \binom{R_i}{k_i} \{F(T) - F(x_i)\}^{k_i} \{1 - F(T)\}^{R_i - k_i} \right\}$$

$$\times \{1 - F(T)\}^{\sum_{i=d+1}^{m}(1+R_i)} \left\{ \prod_{i=d+1}^{m} \sum_{j=i}^{m} (1 + R_j) \prod_{i=d+1}^{m} \frac{f(x_i)}{1 - F(T)} \left\{ \frac{1 - F(x_i)}{1 - F(T)} \right\}^{R_i} \right\}$$

$$= \left\{ \prod_{i=1}^{d} \sum_{j=i}^{m} (1 + R_j) \right\} \{1 - F(T)\}^{\sum_{i=d+1}^{m}(1+R_i)}$$

$$\times \left\{ \prod_{i=1}^{d} \sum_{k_i=0}^{R_i} \binom{R_i}{k_i} F(T)^{k_i+1} \{1 - F(T)\}^{R_i - k_i} \frac{f(x_i)}{F(T)} \left\{ \frac{F(T) - F(x_i)}{F(T)} \right\}^{k_i} \right\}$$

$$\times \left\{ \prod_{i=d+1}^{m} \sum_{j=i}^{m} (1 + R_j) \prod_{i=d+1}^{m} \frac{f(x_i)}{1 - F(T)} \left\{ \frac{1 - F(x_i)}{1 - F(T)} \right\}^{R_i} \right\}$$

$$
\begin{aligned}
= & \left\{\prod_{i=1}^{d}\sum_{j=i}^{m}(1+R_j)\right\}\{1-F(T)\}^{\sum_{i=d+1}^{m}(1+R_i)} \\
& \times\left\{\sum_{k_1=0}^{R_1}\cdots\sum_{k_d=0}^{R_d}\left[\prod_{i=1}^{d}\binom{R_i}{k_i}\frac{F(T)^{k_i+1}\{1-F(T)\}^{R_i-k_i}}{\sum_{j=i}^{d}(1+k_j)}\right]\right. \\
& \left.\times\left[\left\{\prod_{i=1}^{d}\sum_{j=i}^{d}(1+k_j)\right\}\prod_{i=1}^{d}\frac{f(x_i)}{F(T)}\left\{\frac{F(T)-F(x_i)}{F(T)}\right\}^{k_i}\right]\right\} \\
& \times\left\{\prod_{i=d+1}^{m}\sum_{j=i}^{m}(1+R_j)\prod_{i=d+1}^{m}\frac{f(x_i)}{1-F(T)}\left\{\frac{1-F(x_i)}{1-F(T)}\right\}^{R_i}\right\}.
\end{aligned}
$$

This expression reveals that the conditional distributions of the two blocks are as stated in (6). Next, in order to obtain the joint pmf of $K_1,\cdots,K_m$, we have to divide the above expression by $\mathsf{P}(D=d)$ given in (7) and then integrate out the PCOS. If this is carried out, it can be seen that their joint pmf is as stated in the theorem.

In the case of discrete F, consider the integral representation given by Balakrishnan and Dembińska (2008),

$$
g(x_1,\cdots,x_m)\propto\int_{F(x_1-)}^{F(x_1)}\cdots\int_{F(x_m-)}^{F(x_m)}\prod_{i=1}^{m}(1-u_i)^{R_i}\mathrm{d}u_m\cdots\mathrm{d}u_1
$$

for $x_1\leqslant\cdots\leqslant x_m$. The conditional joint pdf of the Type-II PCOS, given $D=d$, is also proportional to the above quantity, i.e.,

$$
\begin{aligned}
g(x_1,\cdots,x_m|D=d)\propto & \int_{F(x_1-)}^{F(x_1)}\cdots\int_{F(x_d-)}^{F(x_d)}\prod_{i=1}^{d}(1-u_i)^{R_i}\mathrm{d}u_d\cdots\mathrm{d}u_1 \\
& \times\int_{F(x_{d+1}-)}^{F(x_{d+1})}\cdots\int_{F(x_m-)}^{F(x_m)}\prod_{i=d+1}^{m}(1-u_i)^{R_i}\mathrm{d}u_m\cdots\mathrm{d}u_{d+1},
\end{aligned}
$$

for $x_1\leqslant\cdots\leqslant x_d\leqslant T<x_{d+1}\leqslant\cdots\leqslant x_m$. As done in the proof of Theorem 1, upon making the transformations $v_i=\frac{u_i}{F(T)}$ for $i=1,\cdots,d$, and $w_i=\frac{u_i-F(T)}{1-F(T)}$ for $i=d+1,\cdots,m$, we obtain

$$
\begin{aligned}
g(x_1,\cdots,x_m|D=d)\propto & \int_{F_1(x_1-)}^{F_1(x_1)}\cdots\int_{F_1(x_d-)}^{F_1(x_d)}\prod_{i=1}^{d}\{1-F(T)v_i\}^{R_i}\mathrm{d}v_d\cdots\mathrm{d}v_1 \\
& \times\int_{F_2(x_{d+1}-)}^{F_2(x_{d+1})}\cdots\int_{F_2(x_m-)}^{F_2(x_m)}\prod_{i=d+1}^{m}\{1-F(T)-[1-F(T)]w_i\}^{R_i}\mathrm{d}w_n\cdots\mathrm{d}w_{d+1},
\end{aligned}
$$

where, as before, $F_1(x)=\frac{F(x)}{F(T)},0<x\leq T$, and $F_2(x)=\frac{F(x)-F(T)}{1-F(T)},T<x$. Upon writing $\{1-F(T)-[1-F(T)]w_i\}^{R_i}=\{1-F(T)\}^{R_i}(1-w_i)^{R_i}$ and

$$
\{1-F(T)v_i\}^{R_i} \quad = \quad \{1-F(T)+F(T)(1-v_i)\}^{R_i}
$$

$$= \sum_{k_i=0}^{R_i} \binom{R_i}{k_i} \{F(T)\}^{k_i} \{1 - F(T)\}^{R_i - k_i} (1 - v_i)^{k_i},$$

we obtain the result. ☐

Remark 4. Observe that the conditional distribution of $\big(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d:m:n}^{(R_1,\cdots,R_m)}\big)$ is the same as the distribution of Type-II PCOS from a sample of *random* size from F but right truncated at T, with a random progressive censoring scheme as well. It is well-known that right truncation of PCOS does not result in PCOS from the corresponding right-truncated distribution [see, for example, Balakrishnan and Aggarwala (2000)] due to the fact that the observations censored before T could have their values to be larger than T. However, their distribution can be expressed as a (multivariate) mixture: K_i represents the (random) number of random variables having their values to be at most T among the R_i observations that are censored at $X_{i:m:n}$.

Remark 5. As in the case of usual OS, in the continuous case, this result can also be established from the Markovian property of the Type-II PCOS. Moreover, by Theorem 3.1 of Bairamov and Özkal (2007), when $R_1 = \cdots = R_m$ the PCOS have the same distribution as the usual OS from the distribution with cdf $G(x) = 1-(1-F(x))^{n/m}$. So, in this special case the result follows directly from Theorem 1.

The result in Theorem 4 can also be generalized to multiple cut-points as follows.

Theorem 5. *Let $X_1, \cdots, X_n$ be iid random variables from some distribution with cdf F and $\big(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)}\big)$ be the Type-II PCOS with progressive censoring scheme $(R_1, \cdots, R_m)$. Let $-\infty \equiv T_0 < T_1 < \cdots < T_p < \infty$ be the cut-points and D_j be the number of Type-II PCOS that lie in the interval $(T_{j-1}, T_j]$ for $j = 1, \cdots, p$. Then, conditional on $(D_1, \cdots, D_p) = (d_1, \cdots, d_p)$, the random vectors*

$$\begin{gathered} \big(X_{1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d_1:m:n}^{(R_1,\cdots,R_m)}\big), \\ \big(X_{d_1+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d_1+d_2:m:n}^{(R_1,\cdots,R_m)}\big), \\ \vdots \\ \big(X_{d_{(p)}+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)}\big) \end{gathered} \tag{8}$$

are mutually independent with

$$\big(X_{d_{(j-1)}+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{d_{(j)}:m:n}^{(R_1,\cdots,R_m)}\big) \overset{\mathrm{d}}{=} \big(V_{1:d_j:d_j+K_{j(d_j)}}^{(j)(K_{j1},\cdots,K_{jd_j})}, \cdots, V_{d_j:d_j:d_j+K_{j(d_j)}}^{(j)(K_{j1},\cdots,K_{jd_j})}\big)$$

for $j = 1, \cdots, p$, and

$$\big(X_{d_{(p)}+1:m:n}^{(R_1,\cdots,R_m)}, \cdots, X_{m:m:n}^{(R_1,\cdots,R_m)}\big) \overset{\mathrm{d}}{=} \big(V_{1:m-d_{(p)}:n-d_{(p)}-R_{(d_{(p)})}}^{(p+1)(R_{d_{(p)}+1},\cdots,R_m)}, \cdots, V_{m-d_{(p)}:m-d_{(p)}:n-d_{(p)}-R_{(d_{(p)})}}^{(p+1)(R_{d_{(p)}+1},\cdots,R_m)}\big),$$

where $d_{(j)} = \sum_{i=0}^{j} d_i$ *with* $d_0 \equiv 0$, $V_1^{(j)}, \cdots, V_{d_j+K_{j(d_j)}}^{(j)}$ *are iid from* F *but doubly truncated in the interval* $(T_{j-1}, T_j]$, $V_1^{(p+1)}, \cdots, V_{n-d_{(p)}-R_{(d_{(p)})}}^{(p+1)}$ *are iid from* F *but left-truncated at* T_p, *and* $K_{j1}, \cdots, K_{jd_j}$ *(for* $i = 1, \ldots, d_j$, $j = 1, \ldots, p$*) are dependent random variables with joint distribution similar to that stated in Theorem 4.*

4 Concomitants of order statistics

Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from a bivariate distribution with joint pdf $f_{X,Y}$, and let the marginal pdf and cdf of X be denoted by f_X and F_X, respectively.

Let $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ be the order statistics of the X-observations and $Y_{[1:n]}, \cdots, Y_{[n:n]}$ be the corresponding concomitants. For a fixed point T, let D denote the number of X's that do not exceed T. We then have the following conditional independence result.

Theorem 6. *Conditional on* $D = d$, *the random vectors* $(Y_{[1:n]}, \cdots, Y_{[d:n]})$ *and* $(Y_{[d+1:n]}, \cdots, Y_{[n:n]})$ *are mutually independent with*

$$\begin{aligned} \left(Y_{[1:n]}, \cdots, Y_{[d:n]}\right) &\overset{\mathrm{d}}{=} \left(U_{[1:d]}, \cdots, U_{[d:d]}\right), \\ \left(Y_{[d+1:n]}, \cdots, Y_{[n:n]}\right) &\overset{\mathrm{d}}{=} \left(Z_{[1:n-d]}, \cdots, Z_{[n-d:n-d]}\right), \end{aligned} \tag{9}$$

where $U_{[1:d]}, \cdots, U_{[d:d]}$ *are the concomitants of order statistics* $V_{1:d}, \cdots, V_{d:d}$ *obtained from a random sample* $(V_1, U_1), \cdots, (V_d, U_d)$ *from* $f_{X,Y}$ *but with* V*'s being right-truncated at* T, *while* $Z_{[1:n-d]}, \cdots, Z_{[n-d:n-d]}$ *are the concomitants of order statistics* $W_{1:n-d}, \cdots, W_{n-d:n-d}$ *obtained from a random sample* $(W_1, Z_1), \cdots, (W_{n-d}, Z_{n-d})$ *from* $f_{X,Y}$ *but with* W*'s being left-truncated at* T.

Proof. The conditional joint pdf of $(X_{1:n}, Y_{[1:n]})$, $\cdots$, $(X_{n:n}, Y_{[n:n]})$, given $D = d$, is given by

$$\begin{aligned} &g((x_1, y_1), \ldots, (x_n, y_n) | D = d) \\ &\qquad = \frac{n! \prod_{i=1}^{n} f_{X,Y}(x_i, y_i)}{\binom{n}{d} \{F_X(T)\}^d \{1 - F_X(T)\}^{n-d}} \, I(x_1 < \cdots < x_d \leqslant T < x_{d+1} < \cdots < x_n) \\ &\qquad = d! \prod_{i=1}^{d} \frac{f_{X,Y}(x_i, y_i)}{F_X(T)} \, I(x_1 < \cdots < x_d \leqslant T) \\ &\qquad\qquad \times (n-d)! \prod_{i=d+1}^{n} \frac{f_{X,Y}(x_i, y_i)}{1 - F_X(T)} \, I(T < x_{d+1} < \cdots < x_n), \end{aligned}$$

which proves the assertion. The corresponding result for discrete X can be established in a manner similar to those presented in the preceding sections. □

5 Conclusions

In this paper, we have established that blocks of ordered data formed by some conditioning events are mutually independent and we have considered for this purpose the usual order statistics, Type-I and Type-II progressively censored order statistics, and concomitants of order statistics. This block independence was a key property in the development of exact conditional inferential procedures based on different forms of censored data [see Childs et al. (2003), Balakrishnan and Iliopoulos (2008), and Balakrishnan, Han and Iliopoulos (2008)] and also for the step-stress tests [see Balakrishnan et al. (2007) and Balakrishnan, Xie and Kundu (2008)]. This conditional independence result has also been implicitly present in some other results such as recurrence relations for order statistics such as those discussed by Govindarajulu (1963) and Balakrishnan, Govindarajulu and Balasubramanian (1993).

In closing, we would like to mention that most of the results presented here can be stated for generalized order statistics (GOS) introduced by Kamps (1995). Recall that the joint pdf of GOS $X_{1:n}^{\widetilde{\gamma}} \leqslant \cdots \leqslant X_{n:n}^{\widetilde{\gamma}}$ from a parent distribution with cdf F is given by

$$\left(\prod_{i=1}^{n} \gamma_i\right) \int_{F(x_1-)}^{F(x_1)} \cdots \int_{F(x_n-)}^{F(x_n)} \prod_{i=1}^{n} (1-u_i)^{\gamma_i - \gamma_{i+1} - 1} \mathrm{d}u_n \cdots \mathrm{d}u_1$$

for $x_1 \leqslant \cdots \leqslant x_n$, where $\gamma_1, \cdots, \gamma_n > 0$ and $\gamma_{n+1} \equiv 0$. Set D to be the number of GOS that do not exceed a fixed point T. From the above integral representation, it can be established that, conditional on $D = d$,

- the random vectors $\widetilde{X}_1 = (X_{1:n}^{\widetilde{\gamma}}, \cdots, X_{d:n}^{\widetilde{\gamma}})$ and $\widetilde{X}_2 = (X_{d+1:n}^{\widetilde{\gamma}}, \cdots, X_{n:n}^{\widetilde{\gamma}})$ are mutually independent, and
- $\widetilde{X}_2 \stackrel{\mathrm{d}}{=} (W_{1:n-d}^{\widetilde{\gamma}_2}, \cdots, W_{n-d:n-d}^{\widetilde{\gamma}_2})$, with $\widetilde{\gamma}_2 = (\gamma_{d+1}, \cdots, \gamma_n)$, are GOS based on F but left-truncated at T.

Note that the conditional distribution of $\widetilde{X}_1$, given $D = d$, can be expressed as a distribution of GOS only when $\gamma_i - \gamma_{i+1} - 1$, $i = 1, \cdots, d$, are non-negative integers. This case, in fact, corresponds to the special case of Type-II PCOS discussed in Section 3.2.

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