SOME NEW ESTIMATORS OF THE BINOMIAL PARAMETER n

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ABSTRACT

On the basis of an observation from the binomial distribution $\mathcal{B}(n, p)$ with known probability of success p, we construct two classes of admissible estimators of the parameter $n \in \{1, 2, \ldots\}$ when the loss function is the squared error. The estimators are either proper Bayes or limits of Bayes estimators.

1. INTRODUCTION

Let X be an observation from the binomial distribution $\mathcal{B}(n, p)$, where the probability of success $p = 1 - q$ is a known constant in $(0, 1)$ and $n \in \mathrm{N}^+ = \{1, 2, \ldots\}$ is an unknown parameter. This context arises in many situations, e.g. estimation of finite population size [cf. Mukhopadhyay (1)] or simple random sampling with replacement. Moreover, a practical application in an animal counting problem has been discussed by Rukhin (2).

In this paper we consider decision theoretic estimation of n under the squared error loss,

$$L_1(\delta, n) = (\delta - n)^2 \ .$$

and the scaled squared error loss,

$$L_2(\delta, n) = n^{-1}(\delta - n)^2 \ .$$

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According to Casella (3) the latter is the most natural loss function for estimating n. Off course, the admissibility problem of the estimators is not affected when L_1 is replaced by L_2. However, in our approach the scaled squared error loss leads to more "ordinary" estimators, that is, estimators of the form

$$\delta(0) = c \quad , \quad \delta(X) = aX + b \ , \ X \geq 1 \ , \tag{1.1}$$

for some constants a, b, $c = c(a, b)$. This is the usual form of estimators of n appeared so far in the literature.

When the parameter space is assumed to be $\mathrm{N} = \{0, 1, 2, \ldots\}$, the natural estimator of n is $\delta^0 = X/p$. Rukhin (2) and Ghosh and Meeden (4) showed that this estimator has optimal properties; it is the only unbiased estimator and is admissible and minimax under quadratic loss. Other relevant work has been done by Feldman and Fox (5), Hamedani and Walter (6), Sadooghi-Alvandi and Parsian (7) (who consider estimation under the asymmetric LINEX loss function) and Yang (8) (who provides a characterization of the admissible linear estimators $aX + b$ under squared error loss).

In the more realistic case of the truncated parameter space, i.e., $n \in \mathrm{N}^+ = \{1, 2, \ldots\}$, the estimator δ^0 is clearly inadmissible under any convex loss. Under squared error, Sadooghi-Alvandi (9) showed that this is also the case for its obvious modification, i.e., the estimator

$$\delta^1(0) = 1 \quad , \quad \delta^1(X) = X/p \ , \ X \geq 1 \ .$$

In the same paper, Sadooghi-Alvandi proved the admissibiliy of the estimator

$$\delta^*(0) = -q/p \log p \quad , \quad \delta^*(X) = X/p \ , \ X \geq 1 \ , \tag{1.2}$$

which is the generalized Bayes estimator of n under the improper prior $\pi(n) = 1/n$, $n = 1, 2, \ldots$ [see also Ghosh and Meeden (4)]. Sadooghi-Alvandi and Parsian (7) obtained analogous results under LINEX loss. Recently, Zou and Wan (10) established an explicit result for estimators of the form (1.1). We restate it here because we will often refer to it in the sequel.

Theorem 1.1. [Zou and Wan (10)] *The estimator*

$$\delta_{a,b}(X) = \begin{cases} \frac{b(a-1)a^b}{a^b-1} & , \quad X = 0 \ , \\ aX + b(a-1) & , \quad X \geq 1 \ , \end{cases} \tag{1.3}$$

is admissible if and only if one of the following three conditions is satisfied: (i) $a = 1$*; (ii)* $1 < a < 1/p$ *and* $b \geq -1$*; (iii)* $a = 1/p$ *and* $-1 \leq b \leq 0$*.*

The above class includes Sadooghi-Alvandi's (9) estimator δ^* in (1.2) as a special case ($a = 1/p$ and $b \to 0$). In their paper, Zou and Wan (10) have also mentioned an estimator resulting from (1.3) by letting $a \to 1$ and $b \to \infty$ in such a way that $b(a-1) \to \log c$ ($c > 1$), that is,

$$\delta_c(X) = \begin{cases} \frac{c \log c}{c-1} & , \quad X = 0 \ , \\ X + \log c & , \quad X \geq 1 \ . \end{cases} \tag{1.4}$$

However, they have not proved its admissibility. Notice that the most of the admissibility results of Zou and Wan described in Theorem 1.1 have been established by assigning negative binomial priors to n truncated in N^+ and obtaining either the corresponding (unique) Bayes estimators or their limits.

In what follows we provide some new estimators of n when the parameter space is N^+. Letting $n-1$ have Poisson or negative binomial prior (rather than n having a truncated one) we obtain the corresponding Bayes estimators with respect to L_1 and L_2. In Section 2 we consider a Poisson prior which results in Bayes estimators of the form

$$T_c(X) = \begin{cases} c + 1 & , \quad X = 0 \ , \\ X + c + \dfrac{c}{X + c} & , \quad X \geq 1 \ . \end{cases} \tag{1.5}$$

These estimators are admissible for any $c \geq 0$. Moreover, under L_2 it will be seen that the resulting Bayes estimator is essentially δ_c in (1.4) and this proves its admissibility. In Section 3 we consider a negative binomial prior and we get under L_1 the class of Bayes estimators

$$T_{r,\theta}(X) = \begin{cases} 1 + r\theta q/(1 - \theta q) & , \quad X = 0 \ , \\ \dfrac{X}{1 - \theta q} + \dfrac{r\theta q}{1 - \theta q} + \dfrac{(r-1)\theta q}{X + (r-1)\theta q} & , \quad X \geq 1 \ , \end{cases} \tag{1.6}$$

where $r > 0$, $0 < \theta < 1$. These estimators are also admissible. Furthermore, we explore the limiting cases. When $r \to 0$, $T_{0,\theta}$ is admissible, whereas when $\theta \to 1$, $T_{r,1}$ is not.

2. BAYES ESTIMATORS UNDER A POISSON PRIOR

Let $X \sim \mathcal{B}(n,p)$ where $p \in (0,1)$ is a known constant and $n \in \mathrm{N}^+ = \{1,2,\ldots\}$ is an unknown parameter. That is, the probability mass function of X is

$$f(x|n) = \frac{n!}{x!(n-x)!}\, p^x q^{n-x} \ , \quad x = 0,1,\ldots,n \ ,$$

where $q = 1-p$. Let the prior distribution of n be

$$\pi_\lambda(n) = e^{-\lambda}\frac{\lambda^{n-1}}{(n-1)!} \ , \quad n = 1,2,\ldots \ , \tag{2.1}$$

i.e., $n-1 \sim \mathcal{P}(\lambda)$ (Poisson with mean λ), $\lambda > 0$. Then, the posterior distribution is

$$\pi^*_\lambda(n|x) = \begin{cases} e^{-\lambda q}\frac{(\lambda q)^{n-1}}{(n-1)!} \ , & n = 1,2,\ldots \ , \quad x = 0 \ , \\ e^{-\lambda q}\frac{n}{x+\lambda q}\frac{(\lambda q)^{n-x}}{(n-x)!} \ , & n = x, x+1,\ldots \ , \quad x \geq 1 \ . \end{cases}$$

It is well–known that the Bayes estimator of n under L_1 is the posterior mean, $\mathrm{E}_\lambda(n|x)$. Obviously, $\mathrm{E}_\lambda(n|0) = 1 + \lambda q$, since conditionally on $X = 0$, $n - 1 \sim \mathcal{P}(\lambda q)$. On the other hand, for $x \geq 1$,

$$\begin{aligned} \mathrm{E}_\lambda(n|x) &= \sum_{n=x}^{\infty} \frac{n^2}{x+\lambda q}\exp(-\lambda q)\frac{(\lambda q)^{n-x}}{(n-x)!} \\ &= \frac{1}{x+\lambda q}\sum_{n=0}^{\infty}(x+n)^2\exp(-\lambda q)\frac{(\lambda q)^n}{n!} \\ &= \frac{1}{x+\lambda q}\{x^2 + 2x\lambda q + \lambda q + \lambda^2 q^2\} \\ &= x + \lambda q + \frac{\lambda q}{x+\lambda q} \ . \end{aligned}$$

It can be seen that the Bayes estimator of n is T_c defined in (1.5) with $c = \lambda q > 0$. Since it is the unique Bayes estimator under (2.1) it is admissible. Taking $c = 0$ we are led to the limiting estimator

$$T_0(0) = 1, \quad T_0(X) = X \ , \ X \geq 1 \ .$$

The admissibility of T_0 follows by Theorem 1.1, since it is the same as $\delta_{1,0}$ in (1.3). Summarizing, we have the following proposition.

Proposition 2.1. *For any $c \geq 0$, the estimator T_c in (1.5) is admissible for n under the squared error loss.*

Consider now Bayesian estimation of n under the loss function L_2. Then, the Bayes estimator is the reciprocal of the posterior expectation of n^{-1}, i.e., $1/\mathrm{E}_\lambda(n^{-1}|X)$. Under (2.1) it holds

$$\begin{aligned}
\mathrm{E}_\lambda(n^{-1}|0) &= \sum_{n=1}^{\infty} \exp(-\lambda q) \frac{(\lambda q)^{n-1}}{n!} \\
&= \frac{1}{\lambda q} \mathrm{P}(Y \geq 1) \quad [\text{where } Y \sim \mathcal{P}(\lambda q)] \\
&= \frac{1 - \exp(-\lambda q)}{\lambda q}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}_\lambda(n^{-1}|x) &= \sum_{n=x}^{\infty} \frac{1}{x + \lambda q} \exp(-\lambda q) \frac{(\lambda q)^{n-x}}{(n-x)!} \\
&= \frac{1}{x + \lambda q} \sum_{n=0}^{\infty} \exp(-\lambda q) \frac{(\lambda q)^n}{n!} \\
&= \frac{1}{x + \lambda q} \ , \ x \geq 1 \ .
\end{aligned}$$

Thus, the Bayes estimator of n with respect to L_2 is given by

$$T'_\lambda(X) = \begin{cases} \frac{\lambda q}{1 - \exp(-\lambda q)} & , \quad X = 0 \ , \\ X + \lambda q & , \quad X \geq 1 \ . \end{cases}$$

Since it is also the unique Bayes estimator and admissibility under L_2 is equivalent to admissibility under L_1, T'_λ is admissible. Setting $\lambda q = \log c$, $c > 1$, it is seen that T'_λ coincides with δ_c in (1.4). Thus,

Proposition 2.2. *For any $c > 1$, the estimator δ_c in (1.4) is admissible for n under the squared error loss.*

3. BAYES ESTIMATORS UNDER A NEGATIVE BINOMIAL PRIOR

Assume now that the prior distribution of n is

$$\pi_{r,\theta}(n) = \frac{\Gamma(r + n - 1)}{\Gamma(r)} (1 - \theta)^r \frac{\theta^{n-1}}{(n-1)!} \ , \quad n = 1, 2, \dots \ , \ r > 0 \ , \ 0 < \theta < 1 \ ,$$

i.e., $n-1 \sim \mathcal{NB}(r, 1-\theta)$ (negative binomial with parameters r, $1-\theta$). Then, the posterior distribution of n is

$$\pi^*_{r,\theta}(n|x) = \begin{cases} (1-\theta q)^r \frac{\Gamma(r+n-1)}{\Gamma(r)} \frac{(\theta q)^{n-1}}{(n-1)!} \ , & n = 1, 2, \dots \ , \quad x = 0 \ , \\ \frac{n(1-\theta q)^{r+x}}{x+(r-1)\theta q} \frac{\Gamma(r+n-1)}{\Gamma(r+x-1)} \frac{(\theta q)^{n-x}}{(n-x)!} \ , & n = x, x+1, \dots \ , \quad x \geq 1 \ . \end{cases} \tag{3.1}$$

The derivation of the posterior means follows. When $x = 0$, we have

$$\mathrm{E}_{r,\theta}(n|0) = 1 + r\theta q/(1-\theta q) \ , \tag{3.2}$$

since conditionally on $X = 0$, $n - 1 \sim \mathcal{NB}(r, 1-\theta q)$. For $x \geq 1$,

$$\begin{aligned} \mathrm{E}_{r,\theta}(n|x) &= \sum_{n=x}^{\infty} \frac{n^2(1-\theta q)^{r+x}}{x+(r-1)\theta q} \frac{\Gamma(r+n-1)}{\Gamma(r+x-1)} \frac{(\theta q)^{n-x}}{(n-x)!} \\ &= \frac{1-\theta q}{x+(r-1)\theta q} \sum_{n=0}^{\infty} (x+n)^2 (1-\theta q)^{r+x-1} \frac{\Gamma(r+n+x-1)}{\Gamma(r+x-1)} \frac{(\theta q)^n}{n!} \\ &= \frac{1-\theta q}{x+(r-1)\theta q} \Bigg\{ x^2 + 2x \frac{(r+x-1)\theta q}{1-\theta q} \\ &\qquad + \frac{(r+x-1)\theta q}{1-\theta q} + \frac{(r+x-1)(r+x)(\theta q)^2}{(1-\theta q)^2} \Bigg\} \\ &= \frac{x}{1-\theta q} + \frac{(r-1)\theta q}{1-\theta q} + \left(\frac{\theta q}{1-\theta q} \right) \frac{x+r-1}{x+(r-1)\theta q} \ . \end{aligned} \tag{3.3}$$

From (3.2), (3.3) we conclude that the Bayes estimator of n is $T_{r,\theta}$ in (1.6). Since $T_{r,\theta}$ is the unique Bayes estimator of n, it is admissible for any $r > 0$, $0 < \theta < 1$. Before stating the corresponding proposition, we explore the limiting cases $r \to 0$ and $\theta \to 1$.

Lemma 3.1. *The estimator*

$$T_{0,\theta}(X) = \frac{X}{1-\theta q} - \frac{\theta q}{X - \theta q}$$

is admissible for n *under the squared error loss for any* $0 < \theta < 1$.

Proof. For convenience, we will supress θ from the subscript. The admissibility of T_0 will be established using a variant of the well–known limiting Bayes method due to Blyth (11) [see also Lehmann and Casella (12), p.380]. Consider first the improper prior

$$\pi_r(n) = \Gamma(r+n-1) \frac{\theta^{n-1}}{(n-1)!} \ , \ n = 1, 2, \dots \ .$$

Since $\Gamma(t)$ attains its minimum for $t \in (0,\infty)$ at $t_0 \approx 1.46163$ [cf. Abramowitz and Stegun (13), p.259], it follows that $\pi_r(n) \geq \Gamma(t_0)\,\theta^{n-1}/(n-1)!$, $n \geq 1$, for any $r > 0$. The posterior distribution of n is $\pi_r^*(n|x) = \pi_{r,\theta}^*(n|x)$ in (3.1), and the Bayes estimator of n with respect to π_r is T_r $(= T_{r,\theta})$. It tends to T_0 as $r \to 0$. Let $I(n \geq x)$ denote the indicator function with the obvious meaning. The marginal distribution of X is

$$m_r(x) = \sum_{n=1}^{\infty} f(x|n)\pi_r(n)I(n \geq x) = \begin{cases} \frac{q\,\Gamma(r)}{(1-\theta q)^r}\ , & x = 0\ , \\ \frac{p^x\theta^{x-1}\Gamma(r+x-1)}{x!(1-\theta q)^{r+x}}\,[x + (r-1)\theta q]\ , & x \geq 1\ . \end{cases}$$

Setting $D_r(n,x) = [T_r(x) - n]^2 - [T_0(x) - n]^2$, the difference of the Bayes risks can be expressed as

$$\begin{aligned} \Delta(r) &= \sum_{n=1}^{\infty}\sum_{x=0}^{n} D_r(n,x)f(x|n)I(n \geq x)\pi_r(n) \\ &= \sum_{n=1}^{\infty} D_r(n,0)f(0|n)\pi_r(n) + \sum_{n=1}^{\infty}\sum_{x=1}^{n} D_r(n,x)f(x|n)I(n \geq x)\pi_r(n) \\ &= \sum_{n=1}^{\infty} D_r(n,0)f(0|n)\pi_r(n) + \sum_{x=1}^{\infty}\sum_{n=x}^{\infty} D_r(n,x)\pi_r^*(n|x)m_r(x)\ . \end{aligned} \tag{3.4}$$

The last equality is obtained by changing the order of summation (this is permitted since the series converges for all $r > 0$) and using the identity $f(x|n)\pi_r(n) = \pi_r^*(n|x)m_r(x)$. The first term in (3.4) equals

$$\sum_{n=1}^{\infty}\left\{\left(n - 1 - \frac{r\theta q}{1-\theta q}\right)^2 - (n-1)^2\right\} q^n\Gamma(r+n-1)\,\frac{\theta^{n-1}}{(n-1)!} =$$

$$\sum_{n=0}^{\infty}\left\{\left(n - \frac{r\theta q}{1-\theta q}\right)^2 - n^2\right\} q^{n+1}\Gamma(r+n)\,\frac{\theta^n}{n!} =$$

$$\left(\frac{r\theta q}{1-\theta q}\right)^2 \sum_{n=0}^{\infty} q^{n+1}\Gamma(r+n)\,\frac{\theta^n}{n!} - \frac{2r\theta q}{1-\theta q}\sum_{n=0}^{\infty} nq^{n+1}\Gamma(r+n)\,\frac{\theta^n}{n!} =$$

$$\frac{qr^2\Gamma(r)}{(1-\theta q)^r}\left(\frac{\theta q}{1-\theta q}\right)^2 - \frac{2qr^2\Gamma(r)}{(1-\theta q)^r}\left(\frac{\theta q}{1-\theta q}\right)^2 =$$

$$-\frac{qr^2\Gamma(r)}{(1-\theta q)^r}\left(\frac{\theta q}{1-\theta q}\right)^2 \longrightarrow 0 \quad \text{as } r \to 0$$

(it holds $r^2\Gamma(r) = r\Gamma(r+1) \to 0$). On the other hand, the inner sum of the second term is equal to

$$\begin{aligned}
&\mathrm{E}_r\left\{[n - T_r(x)|\,x]^2\right\} - \mathrm{E}_r\left\{[n - T_0(x)|\,x]^2\right\} = \\
&\mathrm{Var}_r(n|x) - \left\{\mathrm{Var}_r(n|x) + [\mathrm{E}_r(n|x) - T_0(x)]^2\right\} = \\
&-\{T_r(x) - T_0(x)\}^2 = -r^2\left(\frac{\theta q}{1-\theta q}\right)^2\left[1 + \frac{x(1-\theta q)}{(x-\theta q)(x+(r-1)\theta q)}\right]^2 ,
\end{aligned}$$

since $T_r(x) = \mathrm{E}_r(n|x)$. Hence, the second term in (3.4) equals

$$\begin{aligned}
&-r^2\left(\tfrac{\theta q}{1-\theta q}\right)^2\sum_{x=1}^{\infty}\left[1 + \tfrac{x(1-\theta q)}{(x-\theta q)(x+(r-1)\theta q)}\right]^2 \tfrac{p^x\theta^{x-1}\Gamma(r+x-1)[x+(r-1)\theta q]}{x!(1-\theta q)^{r+x}} = \\
&-r^2\left(\tfrac{\theta q}{1-\theta q}\right)^2\sum_{x=0}^{\infty}\left[1 + \tfrac{(x+1)(1-\theta q)}{(x+1-\theta q)(x+1+(r-1)\theta q)}\right]^2 \tfrac{p^{x+1}\theta^{x}\Gamma(r+x)[x+1+(r-1)\theta q]}{(x+1)!(1-\theta q)^{r+x+1}} = \\
&-\frac{p\,r^2\Gamma(r)(\theta q)^2\,\mathrm{E}_r[h(Y)]}{(1-\theta)^r(1-\theta q)^3} ,
\end{aligned} \tag{3.5}$$

where

$$h(y) = \frac{y+1+(r-1)\theta q}{y+1}\left[1 + \frac{(y+1)(1-\theta q)}{(y+1-\theta q)(y+1+(r-1)\theta q)}\right]^2$$

and $Y \sim \mathcal{NB}[r, (1-\theta)/(1-\theta q)]$. Now, for any $r > 0$, $y \geq 0$,

$$\frac{y+1}{(y+1-\theta q)(y+1+(r-1)\theta q)} \leq \frac{y+1}{(y+1-\theta q)^2} \leq \frac{1}{(1-\theta q)^2} ,$$

and, when $r < 1$, $(y+1+(r-1)\theta q)/(y+1) < 1$. Hence, for $0 < r < 1$, $h(y)$ is bounded above by $[(2-\theta q)/(1-\theta q)]^2$. Therefore, the quantity in (3.5) is in absolute value less than $Cr^2\Gamma(r)$, C being a positive constant not depending on r. Thus, it tends to zero as $r \to 0$. Since both terms in (3.4) tend to zero, $\Delta(r) \to 0$ and consequently, the estimator $T_0 = T_{0,\theta}$ is admissible for any $0 < \theta < 1$.

Lemma 3.2. *For any $r > 0$, the estimator*

$$T_{r,1}(X) = \begin{cases} 1 + rq/p & , \quad X = 0 , \\ X/p + rq/p + \frac{(r-1)q}{X+(r-1)q} & , \quad X \geq 1 , \end{cases}$$

is inadmissible for n under the squared error loss.

Proof. When $r = 1$, the estimator becomes

$$T_{1,1}(X) = \begin{cases} 1/p & , \quad X = 0 \ , \\ X/p + q/p & , \quad X \geq 1 \ , \end{cases}$$

i.e., $\delta_{1/p,1}$ in (1.3), and by Theorem 1.1 it is inadmissible.

Consider now the case $r \neq 1$ (such that $T_{r,1}$ can be written in a unified form) and define the alternative estimator

$$T^*_{r,1,n_0}(X) = \begin{cases} X/p + rq/p + \frac{(r-1)q}{X+(r-1)q} & , \quad 0 \leq X \leq n_0 \ , \\ X/p + \frac{(r-1)q}{X+(r-1)q} & , \quad X > n_0 \ . \end{cases}$$

Obviously, if $n \leq n_0$ the two estimators do not differ and thus, they have equal risks. On the contrary, for $n > n_0$, their risks difference is

$$\begin{aligned} \Delta(n) &= \mathrm{E}_n \left\{ [T_{r,1}(X) - n]^2 I(X > n_0) \right\} - \mathrm{E}_n \left\{ [T_{r,1,n_0}(X) - n]^2 I(X > n_0) \right\} \\ &= \left(\tfrac{rq}{p}\right)^2 \mathrm{P}(X > n_0) + \tfrac{2rq}{p} \mathrm{E}_n \left\{ \left[\tfrac{X}{p} + \tfrac{(r-1)q}{X+(r-1)q} - n \right] I(X > n_0) \right\} = \\ &= \tfrac{rq}{p} \left\{ \tfrac{rq}{p} \mathrm{P}_n(X > n_0) + 2\, \mathrm{E}_n \left[\tfrac{(r-1)q}{X+(r-1)q} \ I(X > n_0) \right] + 2\, \mathrm{E}_n \left[\left(\tfrac{X}{p} - n \right) I(X > n_0) \right] \right\} . \end{aligned}$$

Recall that $\mathrm{E}_n(X/p - n) = 0$. This implies that $\mathrm{E}_n[(X/p - n)I(X > n_0)] \geq 0$ for any $0 \leq n_0 < n$, the inequality being strict when $n_0 \geq 1$.

Suppose first that $r > 1$. Then, by taking $n_0 = 0$ it can be immediately seen that $\Delta(n) > 0$ for all n. For the case $r < 1$, notice that $(r-1)q/(x + (r-1)q)$ is strictly increasing in $x \geq 1$. Therefore, when $n_0 \geq 1$,

$$\begin{aligned} \Delta(n) &> \frac{rq}{p} \left\{ \frac{rq}{p} \mathrm{P}_n(X > n_0) + 2\, \mathrm{E}_n \left[\frac{(r-1)q}{X + (r-1)q} \ I(X > n_0) \right] \right\} \\ &> \frac{rq}{p} \mathrm{P}_n(X > n_0) \left(\frac{rq}{p} + \frac{2(r-1)q}{n_0 + (r-1)q} \right) > 0 \end{aligned}$$

for $n_0 > (1-r)(q + 2p/r)$ (and $n > n_0$). Thus, for suitably chosen n_0, the estimator $T^*_{r,1,n_0}$ dominates $T_{r,1}$ and the statement of the lemma is proved.

Proposition 3.1. *For any $r \geq 0$, $0 \leq \theta < 1$, the estimator $T_{r,\theta}$ in (1.6) is admissible for n under the squared error loss. If $\theta = 1$, it is inadmissible for any $r > 0$.*

The case $r = 0$, $\theta = 1$, corresponding to the estimator

$$T_{0,1}(0) = 1 \;\;,\;\; T_{0,1}(X) = X/p - q/(X-q) \;,\; X \geq 1 \;,$$

is not covered by Proposition 3.1. We feel that $T_{0,1}$ is admissible but we were not able to prove it. In particular, the Blyth's method cannot be applied (this is usually the case when a prior distribution depends on two hyperparameters and the generalized Bayes estimator results by taking both to their extremes).

In closing we comment that the Bayes estimator of n under L_2 has the form

$$T'_{r,\theta}(X) = \begin{cases} \frac{(r-1)\theta q}{(1-\theta q)[1-(1-\theta q)^{r-1}]} & , \quad X = 0 \;, \\ \frac{1}{1-\theta q}\, X + \frac{(r-1)\theta q}{1-\theta q} & , \quad X \geq 1 \;. \end{cases}$$

Setting $a = 1/(1-\theta q)$, $b = r-1$, it can be seen that the above estimator is in fact $\delta_{a,b}$ in (1.3) and thus, its admissibility aspects are covered in detail by Theorem 1.1.

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